Coloured topological operads and moduli spaces of surfaces with multiple boundary curves

Promotionskolloquium

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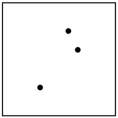
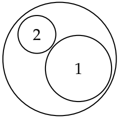
Ingredients

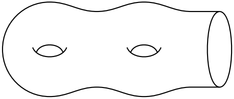
configuration spaces



operads



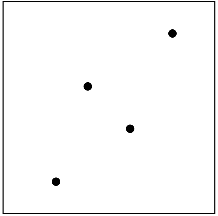
moduli spaces of surfaces



Configuration spaces: Idea

Idea

The *configuration space* $C_r(\mathbb{R}^d)$ describes all possibilities how a collection of r particles can move inside $\mathbb{R}^d$ without any collisions.



$r = 4$

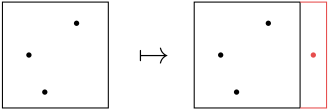
$d = 2$

What can we say about the geometry of these spaces?

Configuration spaces: Stabilisation

Stabilisation

We have maps $C_r(\mathbb{R}^d) \to C_{r+1}(\mathbb{R}^d)$ by adding a new particle:



Stability theorem

The induced map $H_i(C_r \to C_{r+1})$ is an isomorphism for $i \leqslant r/2$.

(Nakaoka 1961, Arnol'd 1969, Segal 1979, Randal-Williams 2013)

Stable configuration space

If we put $C_\infty := \operatorname{colim}(C_0 \to C_1 \to C_2 \to \cdots)$, then

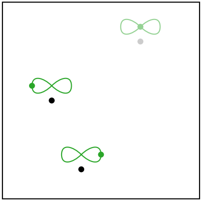
$$H_i(C_r) \cong H_i(C_\infty) \quad \text{for} \quad i \leqslant r/2.$$

Configuration spaces: Labels

Definition

Let X be a based space. We consider $C(\mathbb{R}^d; X)$, where:

- each particle carries a label in X,
- if the label reaches the basepoint, the particle dies.



$X = \infty$

Example

$C(\mathbb{R}^d; S^0) = \coprod_{r \geqslant 0} C_r(\mathbb{R}^d)$.

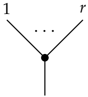
Operads: Idea

Analogy

- If a topological group G acts on a space M, then G *parametrises* unary operations $M \to M$.
- If an operad $\mathcal{O}$ acts on a space M, then $\mathcal{O}$ should parametrise operations $M^r \to M$ for each *arity* $r \geqslant 0$.

Definition

An *operad* $\mathcal{O}$ is an sequence $(\mathcal{O}(0), \mathcal{O}(1), \dots)$ of *operation spaces* (with reasonable extra structure):

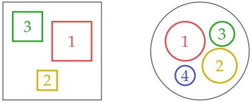


$\in \mathcal{O}(r)$

An *action* of $\mathcal{O}$ on M is a sequence of maps $\mathcal{O}(r) \times M^r \to M$.

Operads: Main example

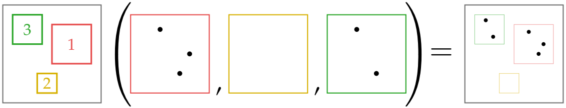
Little d-cubes $\mathscr{C}_d$ / little d-discs $\mathscr{D}_d$ (May 1972)



$\mathscr{C}_2(3)$ $\mathscr{D}_2(4)$

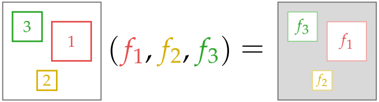
If either $\mathscr{C}_d$ or $\mathscr{D}_d$ acts on M, then M is called an E_d-*algebra*.

$C(\mathbb{R}^d; X)$ is an E_d-algebra



Ω^d-spaces are E_d-algebras

For based Y, let $\Omega^d Y$ be the space of maps $(I^d, \partial I^d) \to (Y, *)$.



Operads: Bar constructions and stable spaces

Recognition principle (May 1972)

If M is an E_d-algebra, then there is a d-fold Bar construction $B^d M$ and $M \to \Omega^d B^d M$ is a group completion.

Group completion theorem (McDuff–Segal 1976)

Let $M = \coprod_{r \geqslant 0} M_r$ *be an* E_d*-algebra. Pick* $e \in M_1$ *and let* $M_\infty := \operatorname{hocolim}(M_0 \xrightarrow{\cdot e} M_1 \xrightarrow{\cdot e} M_2 \to \cdots)$. *Then*

$$M_\infty \simeq_{H_\bullet} \Omega_0^d B^d M.$$

Example (Segal 1973)

$B^d C(\mathbb{R}^d; X) \simeq \Sigma^d X$. Therefore $H_\bullet(C_\infty(\mathbb{R}^d)) \cong H_\bullet(\Omega_0^d S^d)$.

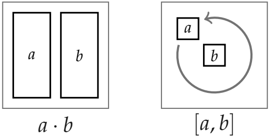
Operads: Homology operations

Homology operations

If M is an $\mathcal{O}$-algebra, then M receives homology operations from $\mathcal{O}$. For example, if M is an E_2-algebra, then $H_\bullet(M)$ is a Gerstenhaber algebra, i.e. there are (among others) operations

$$- \cdot - \colon H_i(M) \otimes H_j(M) \to H_{i+j}(M), \qquad \text{(Pontrjagin product)}$$
$$[-,-] \colon H_i(M) \otimes H_j(M) \to H_{i+j+1}(M), \qquad \text{(Browder bracket)}$$

satisfying various relations (e.g. Jacobi identity).

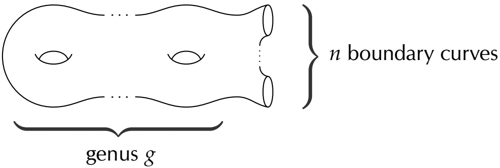


(Pontrjagin 1939, Araki–Kudo 1956, Browder 1960, Dyer–Lashof 1962, Cohen 1976)

Moduli spaces: Idea

Definition

Let $g \geqslant 0$ and $n \geqslant 1$. A *surface of type* $\Sigma_{g,n}$ looks like this:



Moduli spaces

The *moduli space* $\mathfrak{M}_{g,n}$ should classify all $\Sigma_{g,n}$-bundles. Informally,

$$\mathfrak{M}_{g,n} := \{\text{surfaces of type } \Sigma_{g,n}\}.$$

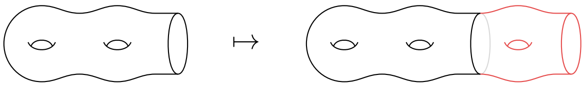
If $p\colon E \to B$ is a $\Sigma_{g,n}$-bundle, then $B \to \mathfrak{M}_{g,n}, x \mapsto p^{-1}(x)$. Thus:

$$H^\bullet(\mathfrak{M}_{g,n}) = \{\text{characteristic classes for } \Sigma_{g,n}\text{-bundles}\}.$$

Moduli spaces: Stabilisation

Genus stabilisation

We have $\mathrm{stab}\colon \mathfrak{M}_{g,1} \to \mathfrak{M}_{g+1,1}$ by attaching a genus-1 surface:



Define $\mathfrak{M}_{\infty,1} \coloneqq \mathrm{colim}(\mathfrak{M}_{0,1} \to \mathfrak{M}_{1,1} \to \mathfrak{M}_{2,1} \to \cdots)$.

Stability theorem

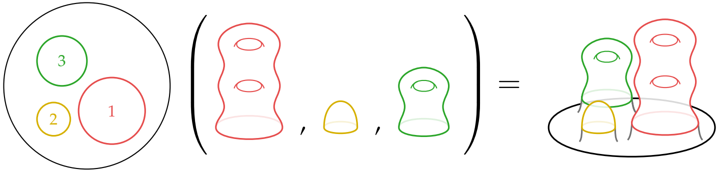
The induced map $H_i(\mathfrak{M}_{g,1}) \to H_i(\mathfrak{M}_{g+1,1})$ is an isomorphism for $i \leqslant 2/3 \cdot (g-1)$.

(Harer 1984, Ivanov 1990, Boldsen 2012, Randal-Williams 2016)

Moduli spaces: Operadic techniques

Construction (Miller 1986, Bödigheimer 1990)

The collection $\coprod_{g \geqslant 0} \mathfrak{M}_{g,1}$ is an E_2-algebra:



Theorem (Tillmann 1997, Madsen–Weiss 2007)

$\Omega B \coprod_{g \geqslant 0} \mathfrak{M}_{g,1} \simeq \Omega^{\infty} \mathbf{MTSO}(2).$

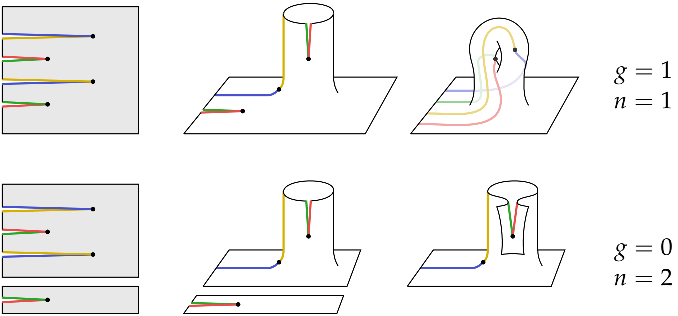
Corollary: Mumford 'conjecture'

$H^{\bullet}(\mathfrak{M}_{\infty,1}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ with $|\kappa_i| = 2i$.

Moduli spaces: Unstable homology

Finite simplicial model (Hilbert 1909, Bödigheimer 1990)

Translate surfaces into configurations of *slits* with gluing data:

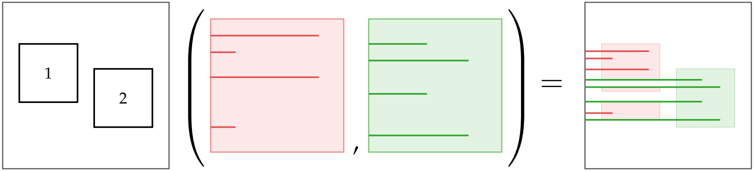


Note that #layers = #curves. Main theorem:

$$\mathfrak{P}_{g,n} := \{\text{slit configurations of type } \Sigma_{g,n}\} \simeq \mathfrak{M}_{g,n}.$$

Moduli spaces: Unstable homology

E_2-action on Bödigheimer's model



„Weltkarte" ('map of the world')

Computer-aided calculations for $H_\bullet(\mathfrak{M}_{g,1})$ for $g \leqslant 3$ using $\mathfrak{P}$.

(Ehrenfried 1998, Abhau 2005, Mehner 2011, Visy 2011, Wang 2011, Boes–Hermann 2014, Boes 2018)

Starting point: Multiple boundary curves

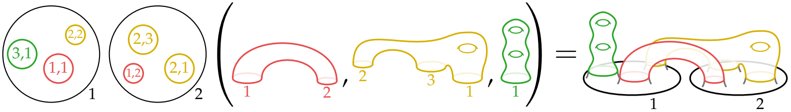
Question

Can we use operadic techniques if we have multiple curves?

Observation 1

We need a *coloured* version $\mathbb{N}(\mathscr{D}_2)$ of $\mathscr{D}_2$ where:

- discs can be placed on *several* large discs,
- several discs can form a common input of higher *multiplicity*.

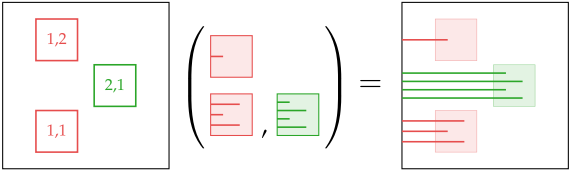


$$\mathbb{N}(\mathscr{D}_2)\binom{2,3,1}{2} \times (\mathfrak{M}_{0,2} \times \mathfrak{M}_{1,3} \times \mathfrak{M}_{2,1}) \to \mathfrak{M}_{4,2}$$

Starting point: Vertical suboperads

Observation 2

If we want to act on $\mathfrak{P}_{\bullet,\bullet}$, then boxes which belong together must share their first coordinate:



Vertical operads (Bödigheimer 2013)

For each decomposition $d = p + q$, we get a suboperad $\mathscr{V}_{p,q} \subseteq \mathbb{N}(\mathscr{C}_d)$ where related boxes share their first p coordinates.

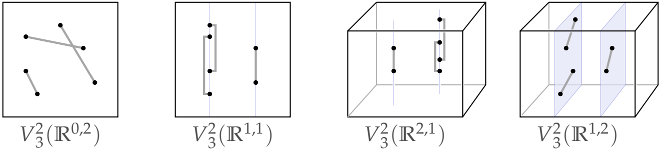
Goal 1: Understand the geometry of $\mathscr{V}_{p,q}$-algebras.
Goal 2: Use the $\mathscr{V}_{1,1}$-action on $\mathfrak{P}_{\bullet,\bullet}$ for homology calculations.

Starting point: Vertical configuration spaces

Observation 3

There are configuration spaces $V_r^k(\mathbb{R}^{p,q})$ related to $\mathscr{V}_{p,q}$:

- r clusters, each of k particles in $\mathbb{R}^{p+q}$,
- all $k \cdot r$ particles are distinct,
- particles from the same cluster share their first p coordinates.



Goal 3: Study these configuration spaces.

(Tran–Palmer 2014, Herberz 2014, Rösner 2014, Latifi 2017, Palmer 2018)

The *Weltkarte*, partially

	0	1	2	3	4
$\mathfrak{M}_{1,1}$	$\mathbb{Z}\langle\mathbf{c}\rangle$	$\mathbb{Z}\langle\mathbf{d}\rangle$			
$\mathfrak{M}_{2,1}$	$\mathbb{Z}\langle\mathbf{c}^2\rangle$	$\mathbb{Z}_{10}\langle\mathbf{cd}\rangle$	$\mathbb{Z}_2\langle\mathbf{d}^2\rangle$	$\mathbb{Z}\langle\lambda\mathbf{s}\rangle \oplus \mathbb{Z}_2\langle T\mathbf{e}\rangle$	$\mathbb{Z}_3\langle\mathbf{w}\rangle \oplus \mathbb{Z}_2\langle ? \rangle$
$\mathfrak{M}_{1,2}$	$\mathbb{Z}\langle\mathbf{c}_2\mathbf{c}\rangle$	$\mathbb{Z}\langle\mathbf{c}_2\mathbf{d}, \mathbf{d}_2\mathbf{c}\rangle$	$\mathbb{Z}\langle\mathbf{d}_2\mathbf{d}\rangle \oplus \mathbb{Z}_2\langle\mathbf{g}_2\rangle$	$\mathbb{Z}_2\langle\hat{T}\mathbf{e}\rangle$	

Table 1. The integral homology of $\mathfrak{M}_{g,n}$

	0	1	2	3	4	5
$\mathfrak{M}_{1,1}$	$\mathbf{c}$	$\mathbf{d}$				
$\mathfrak{M}_{2,1}$	$\mathbf{c}^2$	$\mathbf{cd}$	$\mathbf{d}^2, ?$	$\lambda\mathbf{s}, T\mathbf{e}, Q\mathbf{d}$	$?, TE\mathbf{b}$	$?$
$\mathfrak{M}_{1,2}$	$\mathbf{c}_2\mathbf{c}$	$\mathbf{c}_2\mathbf{d}, \mathbf{d}_2\mathbf{c}$	$\mathbf{d}_2\mathbf{d}, \mathbf{g}_2$	$\hat{T}\mathbf{e}, Q\mathbf{d}_2$	$\hat{T}E\mathbf{b}$	

Table 2. Generators of the $\mathbb{F}_2$-homology of $\mathfrak{M}_{g,n}$