EXERCISE 1.1. The goal of this exercise is to get used to the functor Hom.

A. Bring the following finitely generated abelian groups into the usual form $\mathbb{Z}^r \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_s}$ with $r, s \geqslant 0$ and $k_i \geqslant 1$:
   - $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^3 \oplus \mathbb{Z}_6, \mathbb{Z} \oplus \mathbb{Z}_4)$,
   - $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_{15}^4, \mathbb{Z}_9 \oplus \mathbb{Q})$,
   - $\operatorname{Hom}_{\mathbb{Z}}(A \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Z})$ for some abelian group A.

B. For a commutative ring R, denote by $R[t]$ the polynomial ring in one variable. Let M be an R-module let ϕ be an endomorphism of M. Then M carries the structure of an $R[t]$-module[1] by $\left(\sum_{i=0}^k a_i \cdot t^i\right) \cdot m \coloneqq \sum_{i=0}^k a_i \cdot \phi^i(m)$. Moreover, fix $\lambda \in R$ and consider the $R[t]$-module $R_\lambda \coloneqq R$ with $p \cdot a \coloneqq p(\lambda) \cdot a$. Show that the $R[t]$-module $\operatorname{Hom}_{R[t]}(R_\lambda, M)$ is isomorphic to the eigenspace

$$M_\lambda \coloneqq \{m \in M : \phi(m) = \lambda \cdot m\} \subseteq M.$$

EXERCISE 1.2. The goal of this exercise is to get used to the functor Ext.

A. Bring the following finitely generated abelian groups into the usual form $\mathbb{Z}^r \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_s}$ with $r, s \geqslant 0$ and $k_i \geqslant 1$:
   - $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^3 \oplus \mathbb{Z}_6, \mathbb{Z} \oplus \mathbb{Z}_4)$,
   - $\operatorname{Ext}_{\mathbb{Z}}(\mathbb{Z}_{15}^4, \mathbb{Z}_9 \oplus \mathbb{Q})$.

B. Show that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$ is not isomorphic to $\operatorname{Ext}^1_{\mathbb{Z}_2}(\mathbb{Z}_2, \mathbb{Z}_2)$, where for the second Ext group, we treat both arguments as right $\mathbb{Z}_2$-modules.

C. Using the setting from exercise 1.1:B, show that

$$\operatorname{Ext}^n_{R[t]}(R_\lambda, M) \cong \begin{cases} M/\operatorname{im}(\phi - \lambda \cdot \mathrm{id}_M) & \text{for } n = 1, \\ 0 & \text{for } n \geqslant 2. \end{cases}$$

[1] As $R[t]$ is commutative, we do not have to distinguish between left and right multiplication.

EXERCISE 1.3. Let R, S, T be rings. Let $0 \to C \to C' \to C'' \to 0$ be a short exact sequence of chain complexes of (T, S)-modules that is assumed to be degreewise split. Let N be a (R, S)-module, $a \in H^n(\operatorname{Hom}_S(C, N))$, and $x \in H_{n+1}(C'')$. Show

$$\langle \partial^n a, x \rangle = (-1)^{n+1} \cdot \langle a, \partial_{n+1} x \rangle,$$

where ∂_* and ∂^* are the connecting homomorphisms as in Remark 1.2.6. (*Hint:* Recall the explicit construction of the connecting homomorphism.)

EXERCISE 2.1. Let R be a ring, let N be a left R-module, and let $n \in \mathbb{Z}$.

A. Let $f\colon (X, A) \to (Y, B)$ be a map of pairs such that $H_n(f)$ and $H_{n-1}(f)$ are isomorphisms. Show that $H^n(f; N)$ is an isomorphism of left R-modules.

B. Show that the functor $H^n(-; N)$ satisfies the excision property in the form of Proposition 1.3.5:3.

EXERCISE 2.2. Let $f\colon \mathbb{RP}^2 \to S^2$ be the map that collapses the 1-skeleton of the standard CW structure on $\mathbb{RP}^2$. Recall that $H_2(f; \mathbb{Z}_2)$ is an isomorphism.

A. Show that $H^2(f; \mathbb{Z}_2)\colon H^2(S^2; \mathbb{Z}_2) \to H^2(\mathbb{RP}^2; \mathbb{Z}_2)$ is an isomorphism.

B. Conclude that the Kronecker pairing $\kappa\colon H^2(-; \mathbb{Z}_2) \to \mathrm{Hom}_{\mathbb{Z}}(H_2(-), \mathbb{Z}_2)$ appearing in the geometric UCT admits no natural section (i.e. a natural transformation s going in the other direction that satisfies $\kappa \circ s = \mathrm{id}$).

(This shows that the short exact sequence in the UCT does not split naturally.)

EXERCISE 2.3. Let (X, A) be a pair of spaces, let R and S be rings, let N be an (R, S)-module, and let $n \in \mathbb{Z}$. Assume that *one* of the following assumptions holds:

A. S is a principal ideal domain and $H_{n-1}(X, A; S)$ is a free S-module, or

B. $H_{n-1}(X, A)$ is a free abelian group.

Conclude that the Kronecker pairing $\kappa\colon H^n(X, A; N) \to \mathrm{Hom}_S(H_n(X, A; S), N)$ is an isomorphism of left R-modules. (*Hint:* For B use both the universal coefficient theorem for cohomology and for homology, as well as corollary 1.1.7.)

EXERCISE 3.1. Let R be a commutative ring and X be a space.

A. Let n be odd and let $a \in H^n(X;R)$. Show that $2 \cdot a^2 = 0$, where $a^2 = a \smile a$. Conclude that if 2 is invertible in R, then $a^2 = 0$.

B. Show by example that in A, the assumption that p is odd is necessary. (Do not use $\mathbb{C}\mathrm{P}^d$ or $\mathbb{R}\mathrm{P}^d$, as we have not proved 1.8.15 yet.)

EXERCISE 3.2. Let R be a commutative ring and X be a space. In this exercise, we show that the cup product on $H^\bullet(X;R)$ is unital.

A. Let $*$ be the one-point space and let $\mathrm{pr}_1 \colon * \times X \to X$ and $\mathrm{pr}_2 \colon X \times * \to X$ be the projections. Show that $1_* \times a = \mathrm{pr}_1^* a$ and $a \times 1_* = \mathrm{pr}_2^* a$ for each $a \in H^\bullet(X;R)$. (*Hint:* Acyclic models.)

B. Conclude that indeed $1_X \smile a = a = a \smile 1_X$ holds for each $a \in H^\bullet(X;R)$.

EXERCISE 3.3. Recall that the Alexander–Whitney map AW is only defined up to natural chain homotopy. In this exercise, we discuss an explicit model: For two spaces X and Y, consider aw: $S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$ with

$$\mathrm{aw}(\sigma) \coloneqq \sum_{p+q=n} (\mathrm{pr}_X \circ \sigma)|_{\Delta^p \times \{0\}^q} \otimes (\mathrm{pr}_Y \circ \sigma)|_{\{0\}^p \times \Delta^q}$$

for each n-simplex $\sigma \colon \Delta^n \to X \times Y$, where $\Delta^p \times \{0\}^q$ and $\{0\}^q \times \Delta^q$ are identified with Δ^p and Δ^q, respectively, by forgetting the trivial factor.

A. Show that aw is a chain map and natural in X and Y.

B. Show that after conjugation with the isomorphism $H_0(-) \cong \mathbb{Z}\langle \pi_0(-) \rangle$, the map aw induces the canonical map $\mathbb{Z}\langle \pi_0(X \times Y) \rangle \to \mathbb{Z}\langle \pi_0(X) \rangle \otimes \mathbb{Z}\langle \pi_0(Y) \rangle$ in H_0. Conclude that aw is indeed a model for AW.

EXERCISE 4.1. Let $d \geqslant 0$. Show with the help of cup products that each atlas of the d-torus T^d has at least $d+1$ charts. (Recall that a *chart* of a d-dimensional manifold M is an open subset $U \subseteq M$ together with a homeomorphism $\phi\colon U \to \mathbb{R}^d$. An *atlas* of M is a collection of charts $\{(U_i, \phi_i)\}_{i\in I}$ such that $\bigcup_{i\in I} U_i = M$.)

EXERCISE 4.2. Let R be a commutative ring and let M be a connected d-dimensional manifold (in particular, M is non-empty). Show that each R-orientation μ defines a bijection between the set $R^\times \subset R$ of invertible scalars and the set of R-orientations of M by taking $r \in R^\times$ to $(r \cdot \mu_x)_{x\in M}$.

EXERCISE 4.3. Let $d \geqslant 0$ and let M and N be two compact and connected d-dimensional manifolds, both carrying a ($\mathbb{Z}$-)orientation. We denote their fundamental classes by $[M]$ and $[N]$, respectively. For a map $f\colon M \to N$, we define its *mapping degree* $\deg(f)$ as the unique integer such that $f_*[M] = \deg(f)\cdot[N]$.

A. Show that $\deg(g \circ f) = \deg(g)\cdot\deg(f)$ for two such maps.

B. Show that if $d \geqslant 1$ and f is null-homotopic, then $\deg(f) = 0$.

C. Show that if $\deg(f) \neq 0$, then f is surjective.

D. Show that there is no map $f\colon S^2 \to T^2$ with $\deg(f) \neq 0$.

E. Show that there is a map $f\colon T^2 \to S^2$ with $\deg(f) = \pm 1$.

F. For $a, b, c, d \in \mathbb{Z}$ consider the map $f\colon T^2 \to T^2$ with $(z, w) \mapsto (z^a w^b, z^c w^d)$. Moreover, consider the same orientation on source and target. Show that

$$\deg(f) = \det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

EXERCISE 5.1. Let X be a space and let R be a commutative ring. Use the explicit form of the Alexander–Whitney map from exercise 3.3 to show the following:

A. For $x \in H_\bullet(X;R)$, we have $1_X \frown x = x$.

B. For $x \in H_\bullet(X;R)$ and $a, b \in H^\bullet(X;R)$, we have $(a \smile b) \frown x = a \frown (b \frown x)$.

EXERCISE 5.2. Let M be a compact manifold (in particular, each $H_n(M)$ is finitely generated and $H_n(M) \cong 0$ for all but finitely many n, see lemma 2.4.6). Show that if the dimension of M is odd, then $\chi(M) = 0$.

EXERCISE 5.3. Let M be a compact d-dimensional manifold.

A. Show that if M is orientable, then $H_{d-1}(M)$ is torsion-free.

B. Show that if $H^1(M;\mathbb{Z}_2) \cong 0$, then M is orientable.

(*Remark:* With methods that we did not discuss in the lecture, one can prove B even without the assumption that M is compact.)

BONUS EXERCISE 5.4 (Linking numbers). For the entire exercise, we fix orientations of $S^1 \times S^1$ and S^2. For two disjoint embeddings $f, g\colon S^1 \hookrightarrow \mathbb{R}^3$ of the circle, we define their *linking number* $\mathrm{lk}(f,g) \in \mathbb{Z}$ as the degree of the map

$$h_{f,g}\colon S^1 \times S^1 \to S^2, \quad (z,w) \mapsto \frac{f(z) - g(w)}{\|f(z) - g(w)\|}.$$

A. Show that $\mathrm{lk}(f,g) = -\mathrm{lk}(g,f)$.

B. Show that if f and g are both contained inside $\mathbb{R}^2 \times \{0\}$, then $\mathrm{lk}(f,g) = 0$.

C. Let $f_\bullet, g_\bullet\colon S^1 \times I \to \mathbb{R}^3$ be homotopies such that at each time $s \in I$, the maps $f_s, g_s\colon S^1 \to \mathbb{R}^3$ are disjoint embeddings. Show that $\mathrm{lk}(f_0, g_0) = \mathrm{lk}(f_1, g_1)$.

D. Let $f(x,y) \coloneqq (x, y, 0)$ and $g(x,y) \coloneqq (0, 1+x, y)$. Draw $f(S^1) \cup g(S^1) \subseteq \mathbb{R}^3$ and show $\mathrm{lk}(f,g) = \pm 1$. (*Hint:* Use an argument similar to exercise 4.3E.)

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EXERCISE 6.1. Let M and N be two compact d-dimensional manifolds with R-orientations, and let $\mathrm{PD}_M\colon H^\bullet(M;R) \to H_{d-\bullet}(M;R)$, and similarly PD_N denote their Poincaré duality isomorphisms. For $f\colon M \to N$, we define the *transfer map*

$$f^!\colon H_n(N;R) \xrightarrow{\mathrm{PD}_N^{-1}} H^{d-n}(N;R) \xrightarrow{f^*} H^{d-n}(M;R) \xrightarrow{\mathrm{PD}_M} H_n(M;R).$$

A. Show that if M and N are connected, then $(f_* \circ f^!)(x) = \deg(f) \cdot x$ holds for each class $x \in H_n(N;\mathbb{Z})$.

B. Conclude that there is no map $f\colon \mathbb{C}\mathrm{P}^2 \to S^2 \times S^2$ with $\deg(f) \neq 0$.

C. Recall the genus-g surface Σ_g for each $g \geqslant 0$. Show that there exists a map $f\colon \Sigma_g \to \Sigma_{g'}$ with $\deg(f) \neq 0$ if and only if $g \geqslant g'$.

EXERCISE 6.2. Let R be a commutative ring, let $(I, \leqslant)$ be a directed set, and let $A_\bullet \to B_\bullet \to C_\bullet$ be maps of I-indexed diagrams of R-modules.

A. Show that if each $A_i \to B_i \to C_i$ is exact, then the induced map among colimits $\mathrm{colim}_I(A_\bullet) \to \mathrm{colim}_I(B_\bullet) \to \mathrm{colim}_I(C_\bullet)$ is exact.

B. Show by means of a counterexample that the statement of A is false if we replace ‘directed’ by ‘partially ordered’ in the assumptions. (*Hint:* Pushouts.)

EXERCISE 6.3. Let R be a commutative ring and let M be a non-compact, connected, and R-orientable d-dimesional manifold. Show that $H_d(M;R) \cong 0$.

(*Remark:* With methods that we did not discuss in the lecture, one can prove the statement even without the assumption that M is R-orientable.)

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exercise 7.1 (Eckmann–Hilton argument). Let G be a set and let $\bullet, *: G\times G \to G$ be two multiplications on G, which are assumed to be unital (i.e. both have a neutral element). Moreover, assume that the operations satisfy the *interchange law*

$$(g \bullet g') * (h \bullet h') = (g * h) \bullet (g' * h'). \qquad (g, g', h, h' \in G)$$

Conclude that these two multiplications on G are the same (i.e. $* = \bullet$), associative, and commutative. (*Hint:* Look at the proof of proposition 3.1.8.)

exercise 7.2. Recall the fundamental groupoid $\Pi_1(X)$ assigned to a space X.

- a. For each $f: X \to Y$, construct a functor $\Pi_1(f): \Pi_1(X) \to \Pi_1(Y)$, and show that your construction turns Π_1 into a functor from the category of spaces to the category of (small) categories.
- b. For $f, f': X \to Y$, show that each homotopy between f and f' induces a natural transformation between the functors $\Pi_1(f)$ and $\Pi_1(f')$.
- c. Show that if $f: X \to Y$ is a homotopy equivalence, then $\Pi_1(f)$ is an equivalence of categories.

exercise 7.3. Let X be a space and let $v: I \to X$ be a path from x_0 to x_1. Show that for each $n \geqslant 0$, the following triangle commutes, where $\mathcal{H}$ denotes the Hurewicz homomorphism from section 3.3:

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{\;v_\#\;} & \pi_n(X, x_1) \\ & {\scriptstyle \mathcal{H}_{(X,x_0)}} \searrow \quad \swarrow {\scriptstyle \mathcal{H}_{(X,x_1)}} & \\ & \tilde{H}_n(X). & \end{array}$$

EXERCISE 8.1. Let (X, A, x_0) be a based pair.

A. Show that the map $\pi_1(X, x_0) \to \pi_1(X, A, x_0)$ that is induced by the inclusion $(X, \{x_0\}, x_0) \to (X, A, x_0)$ is injective if and only if its kernel is trivial.

B. Show that the connecting homomorphism $\partial_1 \colon \pi_1(X, A, x_0) \to \pi_0(A, x_0)$ is injective if $\pi_1(X, x_0)$ is trivial.

EXERCISE 8.2. Let (X, A) be a pair and let $v \colon I \to A$ be a path from x_0 to x_1. Recall the map $v_\# \colon \pi_n(X, A, x_0) \to \pi_n(X, A, x_1)$ for $n \geqslant 1$ from construction 3.4.16. Note that v also induces maps on $\pi_n(A, -)$ and $\pi_n(X, -)$ for $n \geqslant 0$, and show that these maps assemble into a morphism of long exact sequences:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \pi_n(A, x_0) & \longrightarrow & \pi_n(X, x_0) & \longrightarrow & \pi_n(X, A, x_0) & \xrightarrow{\partial_n} & \pi_{n-1}(A, x_0) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle v_\#} & & \downarrow{\scriptstyle v_\#} & & \downarrow{\scriptstyle v_\#} & & \downarrow{\scriptstyle v_\#} & & \\ \cdots & \longrightarrow & \pi_n(A, x_1) & \longrightarrow & \pi_n(X, x_1) & \longrightarrow & \pi_n(X, A, x_1) & \xrightarrow[\partial_n]{} & \pi_{n-1}(A, x_1) & \longrightarrow & \cdots \end{array}$$

EXERCISE 8.3. Let $p, q \geqslant 1$ and fix basepoints of S^p and S^q. We want to express $\pi_n(S^p \vee S^q)$ in terms of $\pi_n(S^p)$ and $\pi_n(S^q)$, at least *in a range*:

A. Identifying $S^p \vee S^q$ with the subspace $(S^p \times \{*\}) \cup (\{*\} \times S^q)$ of $S^p \times S^q$, construct an isomorphism $\pi_n(S^p \vee S^q) \cong \pi_n(S^p) \oplus \pi_n(S^q) \oplus \pi_{n+1}(S^p \times S^q, S^p \vee S^q)$ for each $n \geqslant 2$. (*Hint:* Splitting lemma.)

B. Show that $\pi_n(S^p \times S^q, S^p \vee S^q)$ is trivial for $1 \leqslant n \leqslant p + q - 1$. (*Hint:* $S^p \times S^q$ arises from $S^p \vee S^q$ by attaching a $(p+q)$-cell. Now use homotopy excision.)

C. Conclude that $\pi_n(S^p \vee S^q) \cong \pi_n(S^p) \oplus \pi_n(S^q)$ for $2 \leqslant n \leqslant p + q - 2$.

(*Remark:* Part A fails for $n = 1$ as the splitting lemma needs our groups to be abelian; and in fact, $\pi_1(S^1 \vee S^1)$ is non-abelian, even though $\pi_1(S^1) \cong \mathbb{Z}$.)

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EXERCISE 9.1. In this exercise, we show that the inclusion map $\mathbb{C}\mathrm{P}^d \to \mathbb{C}\mathrm{P}^{d+1}$ taking $[z_0, \dots, z_d]$ to $[z_0, \dots, z_d, 0]$ has no left-inverse for $d \geqslant 1$.

A. Show the claim using singular cohomology.

B. Show the claim using homotopy groups.

EXERCISE 9.2. Let $p\colon E \to B$ be a Serre fibration and let $b_0 \in B$ be a basepoint. Put $F \coloneqq p^{-1}(b_0) \subseteq E$ and pick a basepoint $e_0 \in F$. Show that the induced sequence of based maps $\pi_0(F, e_0) \to \pi_0(E, e_0) \to \pi_0(B, b_0)$ is exact at $\pi_0(E, e_0)$.

(*Remark:* This is the last step in the long exact sequence for a Serre fibration.)

EXERCISE 9.3. In this exercise, we construct an example showing that, in constrast to homology, the canonical map $\pi_n(X, A) \to \pi_n(X/A)$ is not an isomorphism in general, even if $A \subseteq X$ is a 'very nice' subspace. Let $d \geqslant 2$.

A. Show that the inclusion of $\mathbb{R}\mathrm{P}^{d-1}$ into $\mathbb{R}\mathrm{P}^d$ induces the trivial map on π_d. (*Hint:* Consider maps of the form $S^{d-1} \to S^d$ is nullhomotopic.)

B. Conclude that $\pi_d(\mathbb{R}\mathrm{P}^d, \mathbb{R}\mathrm{P}^{d-1})$ is isomorphic to $\mathbb{Z}^2$.

C. Conclude that the canonical map $\pi_d(\mathbb{R}\mathrm{P}^d, \mathbb{R}\mathrm{P}^{d-1}) \to \pi_d(\mathbb{R}\mathrm{P}^d/\mathbb{R}\mathrm{P}^{d-1})$ is not an isomorphism.

(*Outlook:* Using homotopy excision, we will see that $\pi_n(X, A) \to \pi_n(X/A)$ is an isomorphism *in a range*, depending on the connectivity of (X, A) and A.)

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EXERCISE 10.1. Let X be a CW complex and $A \subseteq X$ be a contractible subcomplex. Show that the inclusion map $A \to X$ admits a left-inverse. (*Hint:* Use the extension lemma in the form of remark 4.1.5.)

EXERCISE 10.2. Consider the two spaces $X \coloneqq \mathbb{RP}^2$ and $Y \coloneqq \mathbb{RP}^\infty \times S^2$.

A. Show that X and Y have isomorphic homotopy groups in each degree. (Since both X and Y are path-connected, we do not have to specifiy basepoints.)

B. Show that X and Y are not homotopy equivalent.

C. Repeat the subexercises A and B for the spaces $X \coloneqq S^2$ and $Y \coloneqq \mathbb{CP}^\infty \times S^3$.

EXERCISE 10.3. Let X be a path-connected space and let $n \geqslant 1$.

A. Construct a path-connected space $\tau_{\leqslant n}X$ with $\pi_k(\tau_{\leqslant n}X) = 0$ for $k > n$, and a map $\phi_n \colon X \to \tau_{\leqslant n}X$ such that $\pi_k(\phi_n)$ is an isomorphism for each $k \leqslant n$ and each basepoint in X. (*Hint:* Attach cells of dimension at least $n+2$ to X.)

B. Construct a map $\psi_{n+1} \colon \tau_{\leqslant n+1}X \to \tau_{\leqslant n}X$ satisfying $\psi_{n+1} \circ \phi_{n+1} = \phi_n$. (*Hint:* Use once again the extension lemma in the form of remark 4.1.5.)

(*Remark:* The map $\phi_n \colon X \to \tau_{\leqslant n}X$ is called an n^{th} *Postnikov truncation* of X. For example, the inclusion $S^2 \cong \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ is a second Postnikov truncation of S^2. The collection of all ϕ_n and ψ_n is called a *Postnikov tower* for X.)

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EXERCISE 11.1. Prove or disprove the following statements about cofibrations, in which X and Y are spaces and $\imath\colon A \to B$ and $\jmath\colon B \to X$ are maps:

A. The empty map $\varnothing \to X$ is a cofibration.

B. The terminal map $X \to *$ is a cofibration.

C. The inclusion map $X \to X \sqcup Y$ is a cofibration.

D. The projection map $X \times Y \to X$ is a cofibration.

E. If $\imath$ and $\jmath$ are cofibrations, then $\jmath \circ \imath$ is a cofibration.

Which of the statements are true if we replace 'cofibration' by '(Serre) fibration'?

EXERCISE 11.2. Assume that the following diagram is a pushout

$$\begin{array}{ccc} A & \xrightarrow{\;u\;} & B \\ {\scriptstyle \imath}\downarrow & \ulcorner & \downarrow{\scriptstyle \jmath} \\ X & \xrightarrow[\;f\;]{} & Y, \end{array}$$

and assume that $\imath$ is a cofibration.

A. Show that $\jmath$ is a cofibration as well.

B. Show that if u is a homotopy equivalence, then f is a homotopy equivalence.

(*Hint:* If you apply $(-) \times I$ to the above square, the new square is still a pushout.)

C. Formulate and prove a 'dual' version of subexercises A and B, including the words 'fibration' and 'pullback'.

EXERCISE 11.3. Let X be a CW complex.

A. Let $A \subseteq X$ be a contractible subcomplex. Show that the canonical quotient map $X \to X/A$ is a homotopy equivalence.

B. Assume that X is path-connected and 1-dimensional, and has only finitely many cells. Show that X is homotopy equivalent to a bouquet of circles. Express the number of circles in terms of $\chi(X)$.

BONUS EXERCISE 11.4. Assume we knew that $\pi_4(S^3)$ is non-trivial (see outlook 3.7.6). Use this to produce an example showing that even if X arises from A by attaching a d-cell, the induced map $\pi_d(A) \to \pi_d(X)$ need not be injective (in contrast to homology, where this is the case, see the beginning of section 4.2.)

(*Remark:* As one of you pointed out to me, there is an easier example only using non-triviality of $\pi_3(S^2)$, which we have proven in section 3.7.)

EXERCISE 12.1. Let X and Y be homotopy equivalent CW complexes with finitely many cells in each dimension. Moreover, let $d \geqslant 0$ and assume that both X and Y have no cells in dimension $d+1$.

A. Show that $\mathrm{sk}_d X$ and $\mathrm{sk}_d Y$ are homotopy equivalent as well.

B. Give an example showing that the condition on $(d+1)$-cells is necessary.

EXERCISE 12.2. Let X and Y be two spaces. Show that X and Y are weakly homotopy equivalent if and only if they have a common CW approximation, i.e. we find a CW complex Z and weak homotopy equivalences $\varepsilon_X : Z \to X$ and $\varepsilon_Y : Z \to Y$.

EXERCISE 12.3. Let $p: E \to B$ be a fibre bundle with fibre $F \neq \varnothing$, and X a CW complex. Let $d \geqslant 1$ and assume that X is d-connected and $\pi_i(F)$ is trivial for $i \geqslant d$.

A. Show that for each map $f: X \to B$, there is a map $\tilde{f}: X \to E$ with $p \circ \tilde{f} \simeq f$. (*Hint:* Use CW approximation and obstruction theory.)

For the next subexercise, we want to use that Serre fibrations have the homotopy lifting property not only for cubes, but for all CW complexes. Proving this shall not be the main point of this exercise, but you might do it if you want to, using that the pairs $(D^n \times I, S^{n-1} \times I \cup D^n \times \{0\})$ and $(I^n \times I, I^n \times \{0\})$ are homeomorphic.

B. Improve statement A by showing that for each $f: X \to B$ as before, there is a map $\tilde{f}: X \to E$ such that $p \circ \tilde{f}$ is *equal* to f.

C. As an application, show that each fibre bundle $p: E \to B$ with fibre $\mathbb{C}\mathrm{P}^\infty$ and base space B a 3-connected CW complex admits a *section*, i.e. a map $s: B \to E$ that satisfies $p \circ s = \mathrm{id}_B$.

BONUS EXERCISE 12.4. The (ambitious) goal of this exercise is to show the relation $2 \cdot \mathcal{S}[\eta] = 0$ in $\pi_4(S^3)$, where $\eta: S^3 \to S^2$ is the Hopf map and $\mathcal{S}: \pi_3(S^2) \to \pi_4(S^3)$ is the Freudenthal suspension homomorphism.

A. Recall that for $x_0 \in X$, we regarded $\hat{x}_0 := [x_0, \frac{1}{2}] \in SX$ as the basepoint of the suspension. Choosing basepoints $*_n \in S^n$ and fixing homeomorphisms $(SS^n, \hat{*}_n) \cong (S^{n+1}, *_{n+1})$, show that for each based map $\alpha: (S^n, *) \to (X, x_0)$, the class $\mathcal{S}[\alpha]$ coincides with $\pm[S\alpha]$, where $(S\alpha)([z,t]) := [\alpha(z), t]$. (*Hint:* Use the description of relative homotopy groups from remark 4.5.7.)

It is hence goal to show the relation $[S\eta] = -[S\eta]$.

B. Show that for each based map $\alpha\colon (S^n, *_n) \to (S^k, *_k)$ and each based space (X, x_0), the *precomposition map* $(S\alpha)^*\colon \pi_{k+1}(X, x_0) \to \pi_{n+1}(X, x_0)$ given by $[\beta] \mapsto [\beta \circ S\alpha]$ is a homomorphism.

C. Regarding S^3 as a subspace of $\mathbb{C}^2$ and identifying S^2 with $\mathbb{C}\mathrm{P}^1$, consider the complex conjugation maps

$$\kappa_3\colon S^3 \to S^3, \quad (z, w) \mapsto (\bar{z}, \bar{w}),$$
$$\kappa_2\colon S^2 \to S^2, \quad [z{:}w] \mapsto [\bar{z}{:}\bar{w}].$$

If we let $(1, 0) \in S^3$ and $[1{:}0] \in S^2$ be the basepoints, then the maps κ_2 and κ_3 are basepoint-preserving, and the Hopf map $\eta\colon S^3 \to S^2$ is indeed a based map. Show that $[\kappa_3] = [\mathrm{id}_{S^3}]$, $[\kappa_2] = -[\mathrm{id}_{S^2}]$, and $\eta \circ \kappa_3 = \kappa_2 \circ \eta$.

D. Combine B and C in order to show $[S\eta] = -[S\eta]$.

E. Conclude that $\pi_4(S^3)$ is either $\mathbb{Z}_2$ or trivial, and that the suspension homomorphism $S\colon \pi_3(S^2) \to \pi_4(S^3)$ is surjective, but not injective.

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EXERCISE 13.1. Let $d \geqslant 0$. Show that each d-connected and d-dimensional CW complex is contractible. (*Hint:* Use the Hurewicz and the Whitehead theorem.)

EXERCISE 13.2. Let $f\colon (X, A) \to (Y, B)$ be a map of pairs such that for each $x_0 \in A$, the induced maps $\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ and $\pi_n(A, x_0) \to \pi_n(B, f(x_0))$ are bijective for $n \in \{0, 1\}$. Show that the induced map $\pi_1(X, A, x_0) \to \pi_1(Y, B, f(x_0))$ is bijective for all $x_0 \in A$.

(*Remark:* In higher degrees, this is merely an application of the five lemma; yet here we have to deal with the fact that not all constituents are groups.)

EXERCISE 13.3. The goal of this exercise is to determine the isomorphism type of the homotopy group $\pi_2(S^1 \vee S^2)$. (Here we are not in the range of exercise 8.3.)

A. Find a bundle $p\colon E \to S^1 \vee S^2$ with fibre $\mathbb{Z}$ such that E is simply-connected. Here figure 1 might give you some inspiration.

B. Show that the induced map $\pi_2(p)\colon \pi_2(E) \to \pi_2(S^1 \vee S^2)$ is an isomorphism.

C. Show that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}$, and conclude that the range in exercise 8.3C cannot be improved.

(*Remark:* One can see that the action of $\pi_1(S^1 \vee S^2) \cong \mathbb{Z}$ on $\pi_2(S^1 \vee S^2) \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}$ is given by shifting coordinates. This shows that $\pi_2(S^1 \vee S^2)$ is finitely generated as a module over the group ring $\mathbb{Z}[\pi_1(S^1 \vee S^2)]$.)

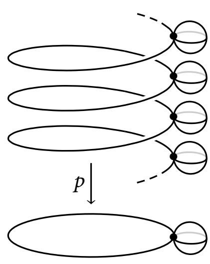

Figure 1. Some inspiration for the fibre bundle $p\colon E \to S^1 \vee S^2$.