VERTICAL CONFIGURATION SPACES & THEIR HOMOLOGICAL STABILITY

(with Andrea Bianchi, Copenhagen)

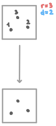
$\boxed{1}$ Reminder: Configuration spaces

Def: Let $\overset{\#\text{points}}{r \geq 0}$, $\overset{\text{dimension}}{d \geq 1}$. Define the ordered configuration space

$$\tilde{C}_r(\mathbb{R}^d) := \{(z_1, \dots, z_r) \in (\mathbb{R}^d)^r ;\ z_i \neq z_j \text{ for } i \neq j\}$$

Then $\mathfrak{S}_r \curvearrowright \tilde{C}_r(\mathbb{R}^d)$ freely by permuting entries
and we define the unordered configuration space

$$C_r(\mathbb{R}^d) := \tilde{C}_r(\mathbb{R}^d) / \mathfrak{S}_r$$



Rmk: Configuration spaces appear at several places:

① For d=2:

$\tilde{C}_r(\mathbb{R}^2)$ is a $K(\pi,1)$ for the pure braid group $\mathcal{PB}_r$,
$C_r(\mathbb{R}^2)$ is a $K(\pi,1)$ for the braid group $\mathcal{B}_r$.

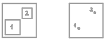
(Fadell–Neuwirth 1962)

For $d \to \infty$

$C_r(\mathbb{R}^\infty)$ is a $K(\pi,1)$ for the symmetric group $\mathfrak{S}_r$.

② The operation spaces of the little d-cubes/discs operad $\mathcal{C}_d$
are ($\mathfrak{S}$-)equivalent to $\tilde{C}_\bullet(\mathbb{R}^d)$

$$\mathcal{C}_d(r) \simeq \tilde{C}_r(\mathbb{R}^d)$$

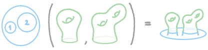


and the free $\mathcal{C}_d$-algebra over a point $*$ is of the form

$$F^{\mathcal{C}_d}(*) = \coprod_{r \geq 0} \mathcal{C}_d(r)/\mathfrak{S}_r \simeq \coprod_{r \geq 0} C_r(\mathbb{R}^d)$$

Thus: Studying $H_*(\tilde{C})$ and $H_*(C)$ is the same as
studying H_*-operations for $\mathcal{C}_d$-algebras.

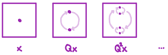
[Side remark: The collection $\coprod_{g \geq 0} \mathcal{M}_{g,1}$ of moduli spaces of surfaces
with a single boundary curve is a $\mathcal{C}_2$-algebra



(Miller 1986,
Bödigheimer 1990)]

Q: What can we say about $H_*(C_r(\mathbb{R}^d))$?

- mod 2: Dyer–Lashof algebra, i.e. a polynomial algebra
generated by classes



(Fuchs 1970,
Cohen 1976)

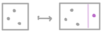
Then $Q^{i_1}x \cdots Q^{i_s}x$
belongs to $H_*(C_r(\mathbb{R}^d))$ if

$$2^{i_1} + \cdots + 2^{i_s} = r$$

- integrally: more difficult, but at least,
the homology groups stabilise.

Def: Have stabilisation maps

$$C_0(\mathbb{R}^d) \hookrightarrow C_1(\mathbb{R}^d) \hookrightarrow C_2(\mathbb{R}^d) \hookrightarrow C_3(\mathbb{R}^d) \hookrightarrow \cdots$$



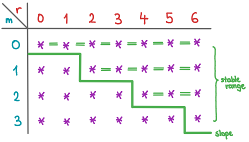
Thm: (Homological stability for configuration spaces)
For each $d \geq 1$, the induced map

$$H_m(C_r(\mathbb{R}^d)) \longrightarrow H_m(C_{r+1}(\mathbb{R}^d))$$

is an isomorphism for $m \leq \frac{r}{2}$.

(Segal 1975, McDuff–Segal 1975, Segal 1979, Randal-Williams 2013)
Arnol'd $d=2$,
Nakaoka $d=\infty$

Rmk: This means that for fixed $d \geq 1$, the table for $H_m(C_r(\mathbb{R}^d))$ looks like this:



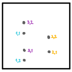
[2] Vertical configuration spaces

Def: Fix $\underbrace{k \geq 1}_{\text{cluster size}}$, $\underbrace{r \geq 0}_{\text{number of clusters}}$, $\underbrace{p \geq 0,\ q \geq 1}_{\text{dimensions}}$

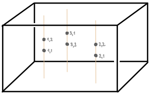
$$\tilde{V}_r^k(\mathbb{R}^{p,q}) = \left\{ \begin{pmatrix} \boxed{\begin{matrix} z_{11} \\ \vdots \\ z_{1k} \end{matrix}} & \begin{matrix} \cdots \\ \\ \cdots \end{matrix} & \begin{matrix} z_{r1} \\ \vdots \\ z_{rk} \end{matrix} \end{pmatrix} \in (\mathbb{R}^{p+q})^{r \times k} ; \begin{array}{l} \bullet \text{ all entries different } z_{i',j'} \neq z_{i,j} \\ \bullet\ \underset{\substack{\text{to the first} \\ p \text{ coordinates}}}{pr}(z_{i,j}) = pr(z_{i,j'}) \end{array} \right\}$$

a cluster

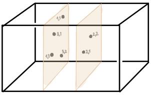
Expl: ① This is a generic point in $\tilde{V}_3^2(\mathbb{R}^{1,1})$



② This is a generic point in $\tilde{V}_3^2(\mathbb{R}^{2,1})$



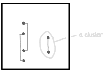
③ This is a generic point in $\tilde{V}_3^2(\mathbb{R}^{1,2})$



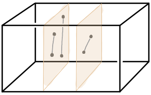
Constr: The symmetric group $\mathfrak{S}_k \wr \mathfrak{S}_r = (\mathfrak{S}_k)^r \rtimes \mathfrak{S}_r$ acts freely on $\tilde{V}_r^k(\mathbb{R}^{p,q})$ and we define

$$V_r^k(\mathbb{R}^{p,q}) := \frac{\tilde{V}_r^k(\mathbb{R}^{p,q})}{\mathfrak{S}_k \wr \mathfrak{S}_r}$$

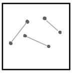
Expl: ① This is a generic point in $V_3^2(\mathbb{R}^{1,1})$



② This is a generic point in $V_3^2(\mathbb{R}^{1,2})$



③ This is a generic point in $V_3^2(\mathbb{R}^{0,2})$



$$V_r^k(\mathbb{R}^{0,q}) = \frac{\tilde{C}_{k\cdot r}(\mathbb{R}^q)}{\mathfrak{S}_k \wr \mathfrak{S}_r}$$

(i.e. a covering of $C_{k\cdot r}(\mathbb{R}^q)$)

Rmk: If $k=1$, then clearly

$$\tilde{V}_r^1(\mathbb{R}^{p,q}) = \tilde{C}_r(\mathbb{R}^{p+q})$$

$$V_r^1(\mathbb{R}^{p,q}) = C_r(\mathbb{R}^{p+q})$$

Why do we care about $\tilde{V}$ and V?

① The spaces $\tilde{V}_r^k(\mathbb{R}^{p,q}) \subseteq \tilde{C}_{r\cdot k}(\mathbb{R}^d)$ are subcomplexes in the cellular decomposition of $\tilde{C}_{r\cdot k}(\mathbb{R}^d)$

(Fox–Neuwirth 1962)

② For $p=0$, the space $V_r^k(\mathbb{R}^{0,q})$ is a configuration space of 0-dim submfds of the form $\{1,\dots,k\}$

(Palmer 2018)

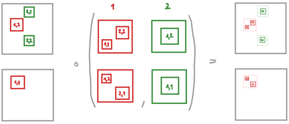
③ There are operads $\mathcal{V}_{p,q}$ whose operation spaces resp. free algebras are modelled by $\tilde{V}$ resp. V, and

- $\mathcal{V}_{p,q}$ is closely related to the extended Swiss cheese operad.

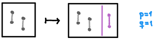
  (Willwacher 2017)

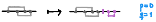
- $\mathcal{V}_{1,1}$ acts on a simplicial model for moduli spaces of surfaces with multiple boundary curves.

  (Bödigheimer 1990,
  Abhau–Bödigheimer–Ehrenfried 2007,
  Boes–Hermann 2014)



Rmk: We have a stabilisation map $V^k_r(\mathbb{R}^{p,q}) \longrightarrow V^k_{r+1}(\mathbb{R}^{p,q})$ by adding a new cluster on the far right side

- 

- 

3 The results

Prop: The induced map in homology

$$H_m(V^k_r(\mathbb{R}^{p,q})) \longrightarrow H_m(V^k_{r+1}(\mathbb{R}^{p,q}))$$

is split injective in all degrees.

McDuff ← Segal

Proof sketch: (similar to Palmer 2017)
We would like to construct a retraction by "forgetting one cluster", but we don't know which one.

⇝ Just "sum" over all choices!

Formally, we get for each $0 \leq i \leq r$ "power-set maps"

$$\begin{tikzcd} \rho_{r,i} : V^k_r(\mathbb{R}^{p,q}) \arrow[r] & SP^{\binom{r}{i}} V^k_i(\mathbb{R}^{p,q}) \\ {[\vec{z}_1,\dots,\vec{z}_r]} \arrow[r, mapsto] & \sum_{\substack{S\subseteq\{1,\dots,r\}\\ \#S=i}} [\vec{z}_j]_{j\in S} \end{tikzcd}$$

$$\xRightarrow{SP^\infty} \quad \begin{tikzcd} SP^\infty V^k_r(\mathbb{R}^{p,q}) \arrow[r] & SP^\infty SP^{\binom{r}{i}} V^k_i(\mathbb{R}^{p,q}) \arrow[r, "\text{“flatten”}"] & SP^\infty V^r_i(\mathbb{R}^{p,q}) \end{tikzcd}$$

$$\xRightarrow{\pi_*} \quad \begin{tikzcd} \tilde{H}_*(V^k_r(\mathbb{R}^{p,q})) \arrow[r, "\tau_{r,i}"'] & \tilde{H}_*(V^r_i(\mathbb{R}^{p,q})) \end{tikzcd}$$

Now (Dold 1962) tells us that such a "system of multiretractions" ensure that the stabilisation is split monic. □

Lem: Let $(0 = A_0 \xrightarrow{s_0} A_1 \xrightarrow{s_1} \dots)$ be a sequence of abelian groups
Assume there are $\tau_{r,i} : A_r \longrightarrow A_i$ for $0 \leq i \leq r$ sth.

- $\tau_{r,r} = \mathrm{id}$
- $\tau_{r,i} - \tau_{r+1,i} \circ s_r : A_r \longrightarrow A_i$ lands in $\mathrm{im}(s_{i-1})$

Then each s_r is split monic.

More difficult is of course:

Thm: (Bianchi–K.) If $(p,q) \neq (0,1)$, then for each $k \geq 1$

$$H_m(V^k_r(\mathbb{R}^{p,q})) \longrightarrow H_m(V^k_{r+1}(\mathbb{R}^{p,q}))$$

is an iso for $m \leq \frac{r}{2}$.

Rmk: ① For $(p,q) = (0,1)$ there is no stability to be expected:

$$\# \pi_0 V^k_r(\mathbb{R}^{0,1}) = \frac{(rk)!}{r!\,(k!)^r} \xrightarrow[r\to\infty]{} \infty \qquad \text{for } k \geq 2.$$

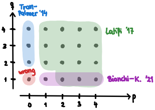
② If $p=0$, we are in the case of configurations of 0-dim submanifolds. This was solved by:

- Tran–Palmer 2014 for $q=2$.
- Palmer 2018 for $q\geq 3$.

③ The case $p+q\geq 3$ was solved by Latifi 2017.

Andrea's and my contribution: A completely different proof for the case $q=1$.

This then covers all cases:



Rest of the talk: Sketch our proof.

4 Interlude: Coloured configurations

(= a special case of Palmer 2017)

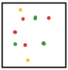
Def: (Coloured configurations)
For a tuple $\alpha=(\alpha_1,\dots,\alpha_h)$ of non-negative integers, we write

- $|\alpha| := \alpha_1+\dots+\alpha_h$ (size)
- $\mathfrak{S}_\alpha := \mathfrak{S}_{\alpha_1}\times\dots\times\mathfrak{S}_{\alpha_h} \subseteq \mathfrak{S}_{|\alpha|}$ (permutations which fix blocks)

Define the coloured configuration space

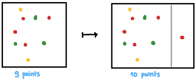
$$C_\alpha(\mathbb{R}^d) := \tilde{C}_{|\alpha|}(\mathbb{R}^d)/\mathfrak{S}_\alpha$$

Elements in $C_\alpha(\mathbb{R}^d)$ are configurations of $|\alpha|$ unordered points in $\mathbb{R}^d$, each one carrying a colour from 1 to h (α_i many with colour i)



◆ = 1 $\alpha_1=4$
♣ = 2 $\alpha_2=3$
♥ = 3 $\alpha_3=2$

Def: We can stabilise by adding a point of the first colour



For $\lambda=(\lambda_1,\dots,\lambda_t)$ and $n\geq|\lambda|$ write

$$\lambda[\underset{\substack{\uparrow\\ \text{total number}\\ \text{of points}}}{n}] := (n-|\lambda|,\lambda_1,\dots,\lambda_t)$$

Then the stabilisation is of the form $C_{\lambda[n]}\longrightarrow C_{\lambda[n+1]}$

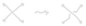
(above: $\lambda=(3,2)$ and we drew $C_{\lambda[9]}\longrightarrow C_{\lambda[10]}$)

- Define an "arc complex" $X_\bullet$ whose s-simplices are tuples $(Z;\gamma_0,\dots,\gamma_s)$ where
  - $Z\in V^k_{r+1}$
  - each γ_i moves one cluster to a standard position
  - $\gamma_i\cap\gamma_j=\emptyset$.

γ_1
γ_2

- Have map $\varepsilon: \|X_\bullet\|\longrightarrow V^k_{r+1} =: X_{-1}$ by remembering only Z. One big lemma shows that the (homotopy) fibre is $(r-1)$-connected; and here we use $p+q\geq 3$

  — essentially because paths can be made disjoint:



  $\Longrightarrow$ By relative Hurewicz, $H_\ell(\mathrm{cone}(\varepsilon))=0$ for $\ell\leq r$.

- Show that $V^k_{r-s}\simeq X_s$. Use this together with the sseq. for the filtration of the augmented cpx $X_\bullet$

$$\underbrace{E^1_{s,t}=H_t(X_s)}_{H_t(V^k_{r-s})} \Longrightarrow \underbrace{H_{s+t+1}(\mathrm{cone}(\varepsilon))}_{\text{0 in low degrees, so everything has to die}} \qquad \begin{array}{l} s\geq -1\\ t\geq 0\end{array}$$

- This implies that the differential

$$d^1_{0,t}: H_t(V^k_r)\longrightarrow H_t(V^k_{r+1})$$

is an iso for $t\leq\frac{r}{2}$.

Thm: (Palmer 2017)

The induced map in homology

$$H_m(C_{\lambda[n]}) \longrightarrow H_m(C_{\lambda[n+1]})$$

is an iso for $m \leq \frac{n-|\lambda|}{2}$.

(In the above case:

$$H_m(C_{\lambda[9]}) \longrightarrow H_m(C_{\lambda[10]})$$

iso for $m \leq \frac{9-(3+2)}{2} = 2$.)

5 Back to vertical configurations: Our proof

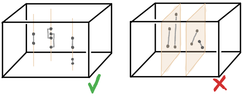
Recall that we want to prove:

Thm: (Bianchi–K.) If $p \geq 1$, then for each $k \geq 1$

$$H_m(V^k_r(\mathbb{R}^{p,1})) \longrightarrow H_m(V^k_{r+1}(\mathbb{R}^{p,1}))$$

is an iso for $m \leq \frac{r}{2}$.

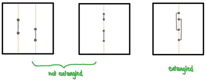
Rmk: $\mathbb{R}^{p,1}$ means that all points of a cluster have to lie on the same vertical line.



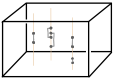
Notation: Fix $k \geq 1$ and $p \geq 1$, and abbreviate $V_r := V^k_r(\mathbb{R}^{p,1})$

STEP 1: The dexterity filtration

- Given a configuration $Z \in V_r$, we call two clusters entangled if their convex hulls intersect
  (implying that they live at the same vertical line)



- Let $1 \leq \delta(Z) \leq r$ be the number of equivalence classes of entangled clusters (= accumulation), called dexterity



$r = 5$, but $\delta(Z) = 4$

- If we define $F_s V_r := \{ Z \in V_r ;\ \delta(Z) \geq r-s \}$, then we have an (open) filtration

$$\emptyset = F_{-1}V_r \subseteq \cdots \subseteq F_{r-1}V_r = V_r$$

  This gives rise to a Leray spectral sequence

$$E(r)^1_{s,t} = H_{s+t}(F_s V_r, F_{s-1} V_r) \Rightarrow H_{s+t}(V_r)$$

- The advantage of filtering V_r in this way is that the stabilisation $V_r \longrightarrow V_{r+1}$ is filtration-preserving

  ⇒ Get maps stab: $F_s V_r \longrightarrow F_s V_{r+1}$

This gives a map of spectral sequences $f: E(r) \longrightarrow E(r+1)$
which has on the 1st page the form

stab.

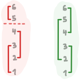
$$f^1_{s,t}: H_{s+t}(F_s V_r, F_{s-1} V_r) \longrightarrow H_{s+t}(F_s V_{r+1}, F_{s-1} V_{r+1})$$

THUS: If we can show that $f^1_{s,t}$ iso for $s+t \leq \frac{r}{2}$,
then the Thm. follows by a comparison argument for SSeq.

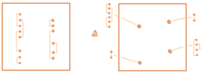
STEP 2: Translating to labelled configurations

- Capture all combinatorial types of "entanglement"

  Let $\mathbb{E}$ be the set of all unordered partitions e of $\{1, \dots, w(e) \cdot k\}$ for some $w(e) \geq 1$, into subsets of size k which are irreducible, i.e. not of the form $e_1 \sqcup e_2$



- Idea: Express a vertical configuration in $F_s V_r$ by a configuration in $\mathbb{R}^{p+1}$ with labels in $\mathbb{E}$:



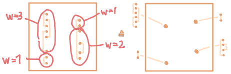
  Need to track how many occurrences of which entanglement type there are.

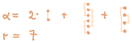
- A distribution is a map $\alpha: \mathbb{E} \longrightarrow \mathbb{N}$ with finite support, also denoted as the formal sum $\sum_{e \in \mathbb{E}} \alpha_e \cdot e$. We let

$$r(\alpha) := \sum_e \alpha_e \cdot w(e) \qquad \text{\#clusters}$$

$$s(\alpha) := \sum_e \alpha_e \cdot (w(e) - 1) \qquad \text{\#clusters} - \text{dexterity}$$

- In our situation above

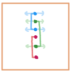




$r = 7$

$s = 3$ (since dexterity 4)

- Idea: In this way, we describe the stratum $F_s V_r \setminus F_{s-1} V_r$.
  In order to describe the pair $(F_s V_r, F_{s-1} V_r)$, we additionally have to track how small perturbations "break" the entanglement:



  For each accumulation, we leave the lowest cluster fixed and the other $w(e)-1$ ones vary inside a small p-disc D^p.

- Given a distribution α, we define the labelled configuration space $D_\alpha(\mathbb{R}^{p+1})$ whose elements are given by:

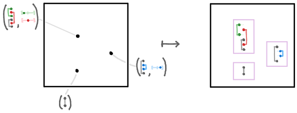
- a configuration of $|\alpha| = \sum_e \alpha_e$ unordered points in $\mathbb{R}^{p+1}$,
- each point carries as label an entanglement type $e \in \mathbb{E}$ and a point in $(D^p)^{w(e)-1}$, such that we have α_e points of type e.

Formally, we put

$$D_\alpha(\mathbb{R}^{p+1}) := \underbrace{\tilde{C}_{|\alpha|}}_{\Sigma \alpha_e}(\mathbb{R}^{p+1}) \times_{\underbrace{\mathfrak{S}_\alpha}_{\prod \mathfrak{S}_{\alpha_e}}} \prod_e \Big(\big(\underbrace{D^p}_{\text{p-dim. disc}}\big)^{w(e)-1}\Big)^{\alpha_e}$$

- For $r := r(\alpha)$ and $s := s(\alpha)$, we have an insertion map

$$\varphi_\alpha : D_\alpha(\mathbb{R}^{p+1}) \longrightarrow F_s V_r$$



$p=1$
$r=6$
$s=3$

dexterity 5, so it even lies in F_1V_6

- Let $D^*_\alpha(\mathbb{R}^{p+1})$ be the subspace of those configurations where at least one label is not in the centre of all discs. Moreover, for fixed $1 \leq s \leq r$, set

$$D_{r,s} := \coprod_{\underbrace{\alpha : (r,s)}_{\substack{r(\alpha)=r \\ s(\alpha)=s}}} D_\alpha(\mathbb{R}^{p+1})$$

$$D^*_{r,s} := \coprod_{\alpha : (r,s)} D^*_\alpha(\mathbb{R}^{p+1})$$

Then we get a map of pairs

$$\varphi_{r,s} = \coprod \varphi_\alpha : (D_{r,s}, D^*_{r,s}) \longrightarrow (F_sV_r, F_{s-1}V_r)$$

- Easy check: $\varphi_{r,s}$ induces an isomorphism

$$H_*(D_{r,s}, D^*_{r,s}) \longrightarrow H_*(F_sV_r, F_{s-1}V_r)$$

this is what we want to study after STEP 1!

Thus, we can instead study the stabilisation for $D_{r,s}$

STEP 3: Stability for $D_{r,s}$

- There is a single entanglement type $e_0 \in \mathbb{E}$ of weight 1. We have a stabilising map

$$D_\alpha(\mathbb{R}^{p+1}) \longrightarrow D_{\alpha + \underbrace{e_0}_{\text{increase } \alpha_{e_0} \text{ by } 1}}(\mathbb{R}^{p+1})$$

by placing a point with label e_0 (and no disc parameter) on the far right. One easily sees:

  - their sum is of the form $D_{r,s} \longrightarrow D_{r+1,s}$ and sends $D^*_{r,s}$ to $D^*_{r+1,s}$.
  - we have a square

$$\begin{tikzcd}
(D_{r,s}, D^*_{r,s}) \arrow[r, "H_*\text{-iso}", "\varphi_{r,s}"'] \arrow[d, "\text{stab}"'] & (F_sV_r, F_{s-1}V_r) \arrow[d, "\text{stab}"] \\
(D_{r+1,s}, D^*_{r+1,s}) \arrow[r, "H_*\text{-iso}", "\varphi_{r+1,s}"'] & (F_sV_{r+1}, F_{s-1}V_{r-1})
\end{tikzcd}$$

Thus, we only have to show that the left stab induces isos in H_m for $m \leq \frac{r}{2}$.

- Recall the coloured configuration spaces $C_\alpha(\mathbb{R}^{p+1})$ from the interlude; here each point carries an entanglement type, but no disc parameter.

  Then we have $p\cdot s$-dimensional disc bundle

  $$D_\alpha(\mathbb{R}^{p+1}) \longrightarrow C_\alpha(\mathbb{R}^{p+1}),$$

  which forgets the disc parameters.

- In this description, $D_\alpha^* = D_\alpha \setminus (0\text{-section})$
  By the Thom isomorphism:

  $$H_m(D_\alpha, D_\alpha^*) \cong H_{m-p\cdot s}(C_\alpha; \underset{\text{sign system}}{\Theta_\alpha}) =: M_{m,\alpha}$$

- Adding a point of the first colour e_0 gives us a map $\gamma_{m,\alpha}: M_{m,\alpha} \longrightarrow M_{m,\alpha+e_0}$ and

$$\begin{tikzcd}
H_m(D_\alpha, D_\alpha^*) \arrow[r, equal] \arrow[d, "\text{stab}"'] & M_{m,\alpha} \arrow[d, "\gamma_{m,\alpha}"] \\
H_m(D_{\alpha+e_0}, D_{\alpha+e_0}^*) \arrow[r, equal] & M_{m,\alpha+e_0}
\end{tikzcd}$$

- Summing over all distributions:

$$\begin{tikzcd}
H_m(D_{r,s}, D_{r,s}^*) \arrow[rr, equal] \arrow[d, "\text{this is what we want}"'] & & \bigoplus\limits_{\alpha:(r,s)} M_{m,\alpha} \arrow[d, "\bigoplus_\alpha \gamma_{m,\alpha} \oplus 0"] \\
H_m(D_{r+1,s}, D_{r+1,s}^*) \arrow[r, equal] & \bigoplus\limits_{\beta:(r+1,s)} M_{m,\beta} \arrow[r, equal] & \bigoplus\limits_{\alpha:(r,s)} M_{m,\alpha+e_0} \oplus \bigoplus\limits_{\substack{\beta:(r+1,s)\\ \beta_{e_0}=0}} M_{m,\beta}
\end{tikzcd}$$

  left to show: for $m \leq \frac{r}{2}$ we have:

  (a) $M_{m,\beta} = 0$ for $\beta_{e_0} = 0$
  (b) $\gamma_{m,\alpha}$ iso

- For (a), we see, since $w(e) \geq 2$ for $\beta_e \neq 0$, and $p \geq 1$,

$$\begin{aligned}
p\cdot s &\geq s \\
&= \textstyle\sum_e \beta_e \cdot (w(e)-1) \\
&\geq \sum_e \beta_e \cdot \left(w(e) - \tfrac{1}{2} w(e)\right) \\
&= \tfrac{1}{2} \cdot \textstyle\sum_e \beta_e \cdot w(e) \\
&= \tfrac{1}{2} \cdot (r+1) \; > m
\end{aligned}$$

  so $M_{m,\beta} = H_{\underbrace{m-p\cdot s}_{<0}}(C_\alpha; \Theta_\alpha) = 0$

- For (b), a similar calculation shows

  $$m \leq \frac{r}{2} \implies m-p\cdot s \leq \frac{\alpha_{e_0}}{2}$$

  Since $w(e) \geq 2$ for $e \neq e_0$, we have

  $$\frac{\alpha_{e_0}}{2} + \sum_{e\neq e_0} \alpha_e \leq \frac{1}{2} \cdot \sum_e \alpha_e \cdot w(e) = \frac{r}{2}.$$

  Now using that $p \geq 1$, we get

  $$m-p\cdot s \leq \frac{r}{2} - \sum_e \alpha_e (w(e)-1) \leq -\frac{r}{2} + \sum_e \alpha_e \leq \frac{\alpha_{e_0}}{2}$$

  Now if we define

  $$\left.\begin{aligned} \lambda &:= (\alpha_e)_{e\neq e_0}, \\ n &:= \sum_e \alpha_e, \end{aligned}\right\} \quad \begin{aligned} &\text{in particular} \\ &\lambda[n] = \alpha \\ &\lambda[n+1] = \alpha + e_0 \end{aligned}$$

  then we have

$$\begin{tikzcd}
M_{m,\alpha} \arrow[r, "\gamma_{m,\alpha}"] \arrow[d, equal] & M_{m,\alpha+e_0} \arrow[d, equal] \\
H_{m-ps}(C_{\lambda[n]}; \Theta_{\lambda[n]}) \arrow[r, "\text{map from Martin's theorem}"'] & H_{m-ps}(C_{\lambda[n+1]}; \Theta_{\lambda[n+1]})
\end{tikzcd}$$

  and the bottom map is (also with the signs!) an iso for

  $$m-p\cdot s \leq \frac{n-|\lambda|}{2} = \frac{\alpha_{e_0}}{2}$$

  □