PARAMETRISED MODULI SPACES OF SURFACES AS INFINITE LOOP SPACES

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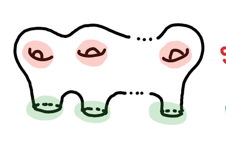
1 Setting and the claim

Def: Let $g \geq 0$ and $n \geq 1$. A surface of type $\Sigma_{g,n}$ looks like this:



Fix such a surface S of type $\Sigma_{g,n}$ and define the mapping class group

$$\Gamma_{g,n} := \pi_0 \mathrm{Diff}^+_\partial(S)$$

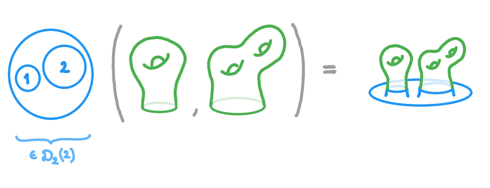
$\{\varphi: S \to S;\ \varphi \text{ or.-pres. diffeo},\ \varphi|_{\partial S} = id\}$

Rmk: One model for $B\Gamma_{g,n}$ is the moduli space

$$\mathcal{M}_{g,n} = \left\{ \begin{array}{l} \text{Riemann surfaces } S \text{ of type } \Sigma_{g,n} \\ + \text{ parametrisation } \{1,\dots,n\} \times S^1 \xrightarrow{\cong} \partial S \end{array} \right\} / \cong$$

> Affine oriented Thom spectrum:
>
> - Let $Gr_2^+(\mathbb{R}^{2+k})$ be the Graßmannian of oriented planes inside $\mathbb{R}^{2+k}$
> - Consider the orthogonal complement of the tautological vector bundle
>
> $$\mathbb{R}^k \to \gamma^\perp_{2,k} \to Gr_2^+(\mathbb{R}^{2+k})$$
>
> and its Thom space
>
> $$MTSO(2)_k := Th(\gamma^\perp_{2,k})$$
>
> - We have a map of $(k+1)$-dim vector bundles
>
> $$\gamma^\perp_{2,k} \oplus \underline{\mathbb{R}} \longrightarrow \gamma^\perp_{2,k+1}$$
>
> This gives us the structure maps
>
> $$\Sigma\, Th(\gamma^\perp_{2,k}) = Th(\gamma^\perp_{2,k} \oplus \underline{\mathbb{R}}) \longrightarrow Th(\gamma^\perp_{2,k+1})$$

Constr: The union $\mathcal{M}_{*,1} = \coprod_{g \geq 0} \mathcal{M}_{g,1}$ is an E_2-algebra by



(Miller 1986, Bödigheimer 1990)

Hence we can form the 1-fold bar construction $B\mathcal{M}_{*,1}$.
Here a miracle happens:

$$\Omega B \mathcal{M}_{*,1} \simeq \Omega^\infty MTSO(2) \qquad \text{(Madsen–Weiss 2007)}$$

(group completion; affine oriented Thom spectrum)

This was the key input for the proof of the Mumford conjecture

$$H^*(\Gamma_{\infty,1}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots] \qquad |\kappa_i| = 2i$$

> By the group completion thm:
>
> $$H_*(\Gamma_{\infty,1} \times \mathbb{Z}) \cong H_*(\mathcal{M}_{*,1})[\pi_0^{-1}]$$
> $$\cong H_*(\Omega B \mathcal{M}_{*,1}) \quad \text{(group completion)}$$
> $$\cong H_*(\Omega^\infty MTSO(2))$$

Our question: Fix a space X. Then the space of $\Sigma_{g,n}$-bundles over X can be described as the mapping space

$$\mathcal{M}^X_{g,n} := \mathrm{map}(X, \mathcal{M}_{g,n}) \qquad \text{"parametrised"}$$

Moreover, $\mathcal{M}^X_{*,1} = \coprod_{g \geq 0} \mathcal{M}^X_{g,1}$ is again an E_2-algebra by pointwise action

⇝ What is $\Omega B \mathcal{M}^X_{*,1}$?

Thm (Bianchi-K.-Reinhold): There is a space $C(X)$ (will be made explicit) such that

$$\Omega B \mathcal{M}^X_{*,1} \simeq \Omega^\infty MTSO(2) \times \Omega^\infty \Sigma^\infty_+ C(X)$$

Rest of the talk: Sketch a proof and describe $C(X)$ for $X = S^1$
(i.e. $\mathcal{M}^X = \Lambda\mathcal{M}$, the free loop space)

2 Proof ingredient 1/2: Centralisers of mapping classes

Rmk: Since $\mathcal{M}_{g,1} \simeq B\Gamma_{g,1}$:

$$\Lambda \mathcal{M}_{*,1} \simeq \coprod_{g \geq 0} \Lambda B\Gamma_{g,1} \simeq \coprod_{g \geq 0} \coprod_{[\varphi] \in \mathrm{Conj}(\Gamma_{g,1})} BZ(\varphi, \Gamma_{g,1})$$

(centraliser of φ inside $\Gamma_{g,1}$; conjugacy classes; need to understand this)

> For a general space X:
> Let $\pi := \pi_1(X)$. Then
>
> $$(B\Gamma)^X = \coprod_{[\varphi] \in \mathrm{Conj}(\pi \to \Gamma)} BZ(\mathrm{im}(\varphi), \Gamma)$$

Constr: Let S be of type $\Sigma_{g,1}$ and let $\varphi \in \Gamma(S) = \Gamma_{g,1}$.
Then there is a maximal subsurface $W \subseteq S$ sth:

- $\partial S \subseteq W$
- W is connected
- "$\varphi|_W = id$"

$r := \#\pi_0 Y$, $k_i := \#\pi_0 \partial Y_i$

> Idea: ① There is a maximal collection $(c_1, \dots, c_m)$ of isotopy classes of disjoint arcs on S which are fixed by φ.
>
> ② Define W to be a tubular neighbourhood of $\partial S \cup c_1 \cup \dots \cup c_m$
>
> Expl: Let S = [surface] and φ = Dehn twist along [curve]
>
> ① A full collection of arcs fixed by φ is given by the single arc c_1
>
> ② Take a tubular nbh of $\partial S \cup c_1$

Another class $\psi \in \Gamma(S)$ commutes with φ if and only if:

① $\psi \simeq \psi_W \sqcup_{\partial^{in} W} (\psi_1 \sqcup \dots \sqcup \psi_r)$

② ψ_i commutes with $\varepsilon_i := \varphi|_{Y_i}$ inside $\Gamma^{\mathfrak{S}}(Y_i)$ (permutation of boundary curves allowed)

Cor: We have a group epimorphism

$$\Gamma^{\mathfrak{S}}(W) \times^{\mathfrak{S}} \prod_{i=1}^r Z(\varepsilon_i, \Gamma^{\mathfrak{S}}(Y_i)) \twoheadrightarrow Z(\varphi, \Gamma_{g,1})$$

$(\psi_W, (\psi_1, \dots, \psi_r)) \mapsto \psi_W \sqcup (\psi_1 \sqcup \dots \sqcup \psi_r)$

and it induces an isomorphism

$$\left(\Gamma^{\mathfrak{S}}(W) \times^{\mathfrak{S}} \prod_{i=1}^r Z(\varepsilon_i, \Gamma^{\mathfrak{S}}(Y_i))\right) \Big/ \prod_i \mathbb{Z}^{k_i} \cong Z(\varphi, \Gamma_{g,1})$$

((D, D^{-1}) with D a Dehn twist along a curve of $\partial^{in} W$; $k_i := \#\pi_0 \partial Y_i$)

> Here I cheated a bit:
> If Y_i and Y_j are of the same type, then it may happen that φ restricts to a diffeo $\varphi_{ij}: Y_i \to Y_j$ and
>
> $$\begin{array}{ccc} Y_i & \xrightarrow{\varphi_{ij}} & Y_j \\ \varepsilon_i \downarrow & & \downarrow \varepsilon_j \\ Y_i & \xrightarrow{\varphi_{ij}} & Y_j \end{array}$$

Passing to classifying spaces (writing $T^n \simeq B\mathbb{Z}^n$ for the n-torus):

$$BZ(\varphi, \Gamma_{g,1}) \simeq \mathcal{M}(W) \times_{\prod_i T^{k_i} \rtimes \mathfrak{S}_{k_i}} \prod_{i=1}^r BZ(\varepsilon_i, \Gamma(Y_i))$$

("irreducible")

> $T^{k_i} \rtimes \mathfrak{S}_{k_i} \curvearrowright \mathcal{M}(W)$ by
>
> - rotating
> - permuting
>
> the boundary parametrisations $S^1 \times \{1, \dots, k_i\} \to \partial W$
>
> and the same holds for $BZ(\varepsilon_i, \Gamma(Y_i)) \subseteq \Lambda\mathcal{M}(Y_i)$

Summing over all conjugacy classes $[\varphi]$

$$\Lambda \mathcal{M}_{*,1} \simeq \coprod_{r \geq 0} \left(\coprod_{k_1, \dots, k_r \geq 1} \mathcal{M}\binom{k_1, \dots, k_r}{1} \times_{\prod_i T^{k_i} \rtimes \mathfrak{S}_{k_i}} \prod_{i=1}^r \underbrace{\coprod_{g \geq 0} \coprod_{[\varepsilon] \text{ irred.}} BZ(\varepsilon, \Gamma_{g,k_i})}_{C_{k_i}} \right) \Big/ \mathfrak{S}_r$$

($k_1, \dots, k_r$: the incoming curves; 1: outgoing curve; $\mathcal{R}_{k_i}$; number $Y_1, \dots, Y_r$)

This looks like a "free algebra" over $C_\bullet$ for an operad $\mathcal{M}$.

3 An operadic reformulation

Def: Let N be a set. An N-coloured operad is a collection of "operation spaces"

$$\mathcal{O}\binom{k_1, \dots, k_r}{n}$$

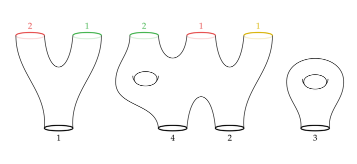
$r \geq 0$: number of inputs; $k_1, \dots, k_r \in N$: the input colours; $n \in N$: the output colour

An $\mathcal{O}$-algebra is a family $A = (A_n)_{n \in N}$ (n^{th} level) together with maps

$$\mathcal{O}\binom{k_1, \dots, k_r}{n} \times \prod_{i=1}^r A_{k_i} \longrightarrow A_n$$

Expl: For each operad $\mathcal{O}$, the family $(\mathcal{O}\binom{}{n})_{n \in N}$ (no input) is the initial $\mathcal{O}$-algebra.

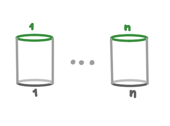
Expl: (Tillmann 2000: Coloured surface operad $\mathcal{M}$)
$N = \{1, 2, 3, \dots\}$ and e.g.



$$\in \mathcal{M}\binom{2, 1, 2}{4}$$

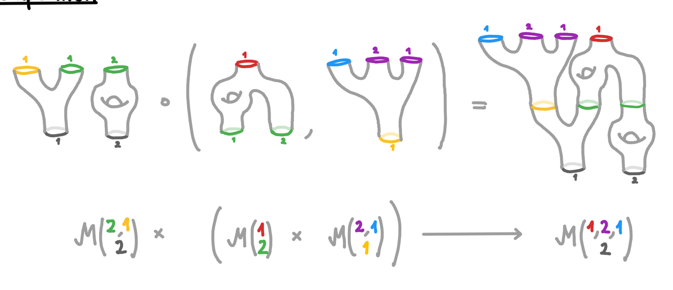
The operadic structure is defined as follows:

① Identities



$$\mathbb{1}_n = \dots \in \mathcal{M}\binom{n}{n}$$

② Composition

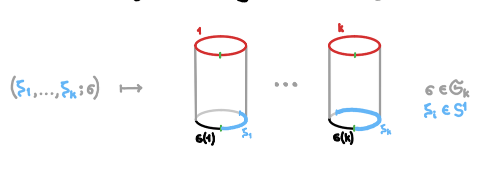


$$\mathcal{M}\binom{2,1}{2} \times \left(\mathcal{M}\binom{1}{2} \times \mathcal{M}\binom{2,1}{1}\right) \longrightarrow \mathcal{M}\binom{1,2,1}{2}$$

Constr: There is a suboperad $\mathcal{R}$ given by twisted tori

$$\mathcal{R}\binom{k}{k} = T^k \rtimes \mathfrak{S}_k \qquad \text{(and no other operations)}$$

and $\mathcal{R} \hookrightarrow \mathcal{M}$ by permuting and rotating ∂-curves



$\mathcal{R}$ is just a sequence of compact Lie groups and

$$[\mathcal{R}\text{-alg}] = \left[\begin{array}{l} \text{sequences of spaces } (Z_k)_{k \geq 1} \\ \text{together with } \mathcal{R}_k \curvearrowright Z_k \end{array} \right]$$

Constr: If $\mathcal{P} \hookrightarrow \mathcal{O}$ is a suboperad, then we have an adjunction

$$F^{\mathcal{O}}_{\mathcal{P}}: [\mathcal{P}\text{-alg}] \rightleftarrows [\mathcal{O}\text{-alg}] : U^{\mathcal{O}}_{\mathcal{P}}$$

("$\mathcal{O} \circ_{\mathcal{P}} -$"; restrict action)

Rmk: The above result says: for $C := (C_k)_{k \geq 1}$, we have

$$\Lambda \mathcal{M}_{*,1} \simeq F^{\mathcal{M}}_{\mathcal{R}}(C)_1$$

$$\Lambda \mathcal{M}_{*,1} \simeq \coprod_{r \geq 0} \left(\coprod_{k_1, \dots, k_r \geq 1} \mathcal{M}\binom{k_1, \dots, k_r}{1} \times_{\prod_i \mathcal{R}_{k_i}} \prod_{i=1}^r C_{k_i} \right) \Big/ \mathfrak{S}_r$$

4 Proof ingredient 2/2: Operads with homological stability

(goes back to / extends: Basterra–Bobkova–Tillmann–Ponto–Yeakel 2017)

General problem: For "nice" N-coloured operads $\mathcal{O}$ containing a family $\mathcal{G} = (G_k)_{k \in N}$ of groups, and a $\mathcal{G}$-algebra Z, describe $\Omega B\, F^{\mathcal{O}}_{\mathcal{G}}(Z)_n$ (n^{th} level)

⇝ Such an operad $\mathcal{O}$ at least needs to know how to group complete each level of its algebras!

Def: An N-coloured operad over $\mathcal{D}_2$ is a map

$$\iota: \mathcal{D}_2 \otimes N \longrightarrow \mathcal{O}$$

(one copy of the little 2-discs operad for each colour $n \in N$.)

$$(\mathcal{D}_2 \otimes N)\binom{n, \dots, n}{n} = \mathcal{D}_2(r)$$

> In principle, each A_∞-operad and a map satisfying the 'weak homotopy commutativity condition' would work.

This gives $\mathcal{O}$ a lot of extra structure:

① If $A = (A_n)_{n \in N}$ is an $\mathcal{O}$-algebra, then each A_n is an E_2-algebra.

- $\pi_0(A_n)$ is an abelian monoid,
- we can consider $\Omega B A_n$

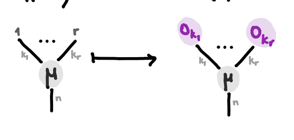
② For each colour $n \in N$, we have an operation

$$O_n := \iota(\bigcirc \otimes n) \in \mathcal{O}\binom{}{n}$$

(empty conf. in $\mathcal{D}_2(0)$)

These can be used to cap inputs:

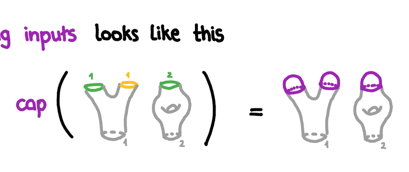
$$\mathrm{cap}: \mathcal{O}\binom{k_1, \dots, k_r}{n} \longrightarrow \mathcal{O}\binom{}{n}$$



Expl: $\mathcal{M}$ is an operad over $\mathcal{D}_2$ by

$$\iota(\text{(1)(2)} \otimes 3) = \dots \in \mathcal{M}\binom{3,3}{3}$$

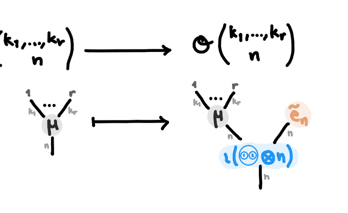
Capping inputs looks like this



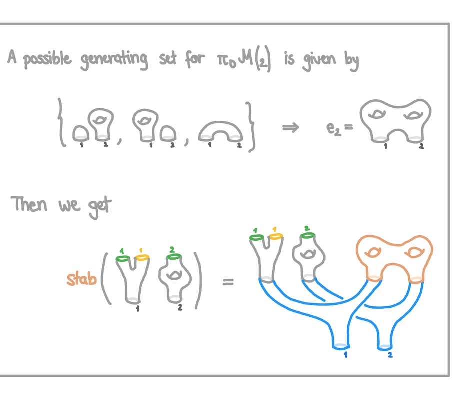
Constr: Let $\mathcal{O}$ be an N-coloured operad over $\mathcal{D}_2$ such that each $\pi_0 \mathcal{O}\binom{}{n}$ is finitely generated as an abelian monoid.

- Choose a finite generating set $E_n \subseteq \pi_0 \mathcal{O}\binom{}{n}$, define $e_n := \sum_{e \in E_n} e$ and choose a propagator $\tilde{e}_n \in \mathcal{O}\binom{}{n}_{e_n}$ (e_n-path component)
- Fix a binary operation $\text{(o)(o)} \in \mathcal{D}_2(2)$. Then we get a stabilisation map

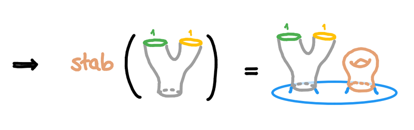
$$\mathrm{stab}: \mathcal{O}\binom{k_1, \dots, k_r}{n} \longrightarrow \mathcal{O}\binom{k_1, \dots, k_r}{n}$$



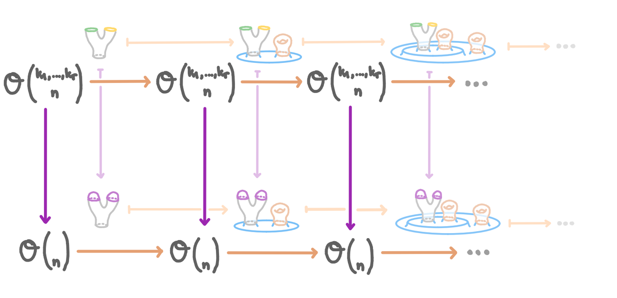
"$+ \tilde{e}_n$"

> A possible generating set for $\pi_0 \mathcal{M}\binom{}{2}$ is given by
>
> 
>
> Then we get

Expl: We have $\mathcal{M}\binom{}{1} = \coprod_{g \geq 0} \mathcal{M}_{g,1}$, so $\pi_0 \mathcal{M}\binom{}{1} \cong \mathbb{N}$. Pick [surface] $\in \mathcal{M}_{1,1} = \mathcal{M}\binom{}{1}_1$



Constr: We get a ladder



$\mathcal{O}$ is called operad with homological stability (OHS) if hocolim(↓) is a H_*-equivalence for each $k_1, \dots, k_r, n \in N$.

Expl: $\mathcal{M}$ is an OHS (essentially Harer 1985)

Thm: (Basterra–Bobkova–Ponto–Tillmann–Yeakel 2017, B.-K.-R.)
If $\mathcal{O}$ is an N-coloured OHS and $\mathcal{G} = (G_k)_{k \in N} \hookrightarrow \mathcal{O}$ is a family of compact Lie groups, then for each $\mathcal{G}$-alg. $Z = (Z_k)_{k \in N}$:

$$\Omega B\, F^{\mathcal{O}}_{\mathcal{G}}(Z)_n \simeq \Omega B\, \mathcal{O}\binom{}{n} \times \Omega^\infty \Sigma^\infty_+ \left(\coprod_{k \in N} Z_k /\!\!/ G_k \right)$$

("forget Z"; "forget the operad: free E_∞-alg. over Z"; homotopy quotient)

Now we have everything together:

$$\Omega B \Lambda \mathcal{M}_{*,1} \simeq \Omega B\, F^{\mathcal{M}}_{\mathcal{R}}(C)_1$$

$$\simeq \Omega B\, \underbrace{\mathcal{M}\binom{}{1}}_{= \mathcal{M}_{*,1}} \times \Omega^\infty \Sigma^\infty_+ \underbrace{\coprod_{k \geq 1} C_k /\!\!/ T^k \rtimes \mathfrak{S}_k}_{=: C(S^1)}$$

$$\simeq \Omega^\infty MTSO(2) \times \Omega^\infty \Sigma^\infty_+ C(S^1)$$

> Similarly can consider the top. category
>
> $$\mathcal{C}ob^{\theta}_d(X), \quad d \geq 1 \qquad \text{(Raptis–Steimle 2017)}$$
>
> (tangential structure, e.g. orientation)
>
> of d-dim. cobordisms over X. What is $B\mathcal{C}ob^{\theta}_d(X)$?
> In the classical case:
>
> - "Positive boundary subcategory" $\mathcal{C}ob_{d,\partial} \hookrightarrow \mathcal{C}ob_d$ induces eq.
>
> $$B\mathcal{C}ob_{d,\partial} \xrightarrow{\sim} B\mathcal{C}ob_d.$$
>
> - Have $\mathcal{C}ob_{2,\partial}(\emptyset, S^1) = \mathcal{M}_{*,1}$. Get map
>
> $$\mathcal{C}ob_{2,\partial}(\emptyset, S^1) \longrightarrow \Omega_{\emptyset, S^1} B\mathcal{C}ob_{2,\partial} \simeq \Omega B\mathcal{C}ob_{2,\partial}$$
>
> (paths from the vertex $\emptyset$ to the vertex S^1)
>
> Can show (Galatius–Madsen–Tillmann–Weiss 2009)
> This is a group completion.