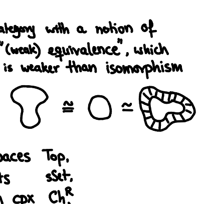
# INTRODUCTION TO MODEL CATEGORIES

(main reference: Hovey "Model categories", §1)

## 1 Motivation and first definitions

"Homotopy theory" = Category with a notion of "(weak) equivalence", which is weaker than isomorphism



**Ex:**
- Topological spaces Top,
- Simplicial sets sSet,
- Positive chain cpx $Ch^R_+$
- Small categories Cat.

**Goal:** Understand "objects up to equivalence"

⇝ Make weak eq. isos by adding formal inverses. (= localise at equivalences)

**Constr:** (Homotopy category) Let $\mathcal{W} \subseteq Mor(\mathcal{C})$ a class of morphisms.
Define the homotopy cat. $Ho(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$:
- objects: objects of $\mathcal{C}$
- morph: $X \to \cdot \to \cdot \leftarrow \cdot \to \cdot \leftarrow \cdot \to Y$ / where $\to$ morph in $\mathcal{C}$, $\leftarrow$ weak equivalence

The canonical $\gamma: \mathcal{C} \to Ho(\mathcal{C})$ satisfies:
- For $w \in \mathcal{W}$, $\gamma(w)$ iso.
- For $F: \mathcal{C} \to \mathcal{D}$ with $Fw$ iso for all $w \in \mathcal{W}$,

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \gamma\downarrow & \nearrow_{\exists! \tilde{F}} & \\ Ho(\mathcal{C}) & & \end{array}$$

**Probl:** $Ho(\mathcal{C})$ very large! (in general not locally small)
⟶ Need better description.
⟶ Need more information about $\mathcal{C}$.

**Idea:** Introduce two further classes of morph in $\mathcal{C}$:

$\mathcal{C}of$ = cofibrations ("nice injections")
$Fib$ = fibrations ("nice surjections")

be careful: Serre fibrations are only surjective onto components which they hit.

which behave nicely wrt. equivalences. These will give rise to classes of "nice" objects:
- cofibrant objects ($\emptyset \to X$ cofib), ($\emptyset$ initial obj.)
- fibrant objects ($X \to *$ fib). ($*$ terminal obj.)

and it will be easier to understand them up to equiv.

**Ex:** In our examples, we will have

| | $\mathcal{W}$ | $\mathcal{C}of$ | Fib | cofibrant | fibrant | |
|---|---|---|---|---|---|---|
| Top | $\pi_*$-iso | generalised CW pairs | Serre fibrations | generalised CW cpx. | all | "Quillen model structure" |
| sSet | $\lvert\cdot\rvert$ $\pi_*$-iso | levelwise injections | Kan fibrations | all | Kan cpxes | "Kan-Quillen model structure" |
| $Ch^R$ | $H_*$-iso ("quasi-iso") | ⊆ levelwise split monos | levelwise epis | ⊇ bounded below levelwise projectives | all | "projective model structure" |
| | | levelwise monos | ⊆ levelwise split epis | all | ⊇ bounded above levelwise injectives | "injective model structure" |

There are also other famous model structures on top. spaces, e.g.
- Hurewicz m.s.
- mixed m.s.

## 2 Axioms for a model cat

**Def:** $f: X \to Y$ is a retract of $f': X' \to Y'$ if there is a diagram

$$\begin{array}{ccccc} X & \xrightarrow{a} & X' & \xrightarrow{a'} & X \\ f\downarrow & & f'\downarrow & & f\downarrow \\ Y & \xrightarrow{b} & Y' & \xrightarrow{b'} & Y \end{array}$$

s.th. $a'a = id_X$ and $b'b = id_Y$

**Def:** A functorial factorisation is a pair of functors $\alpha, \beta: Mor(\mathcal{C}) \to Mor(\mathcal{C})$ such that for all $f: X \to Y$, we have

$$X \xrightarrow{\alpha(f)} S(f) \xrightarrow{\beta(f)} Y, \quad f = \beta(f)\alpha(f)$$

($dom\, \beta(f) = codom\, \alpha(f)$)

Def: (Morphism cat). Let $\mathcal{C}$ cat. Have cat $Mor(\mathcal{C})$:
- objects: Morphisms $f: X \to Y$
- morphisms: Squares $\begin{array}{ccc} X & \xrightarrow{a} & X' \\ f\downarrow & & \downarrow f' \\ Y & \to & Y' \end{array}$

**Def:** (Lifting property) Consider
$i: A \to B$
$p: X \to Y$
If for each square, the following lift exists

$$\begin{array}{ccc} A & \to & X \\ i\downarrow & \nearrow & \downarrow p \\ B & \to & Y \end{array}$$

then say that:
- $i$ LLP wrt. $p$
- $p$ RLP wrt. $i$

**Expl:** If $\mathcal{C}$ has initial obj. $\emptyset$, then
B projective ⟺ $\emptyset \to B$ has LLP wrt. each epi $X \twoheadrightarrow Y$

**Def:** (Model categories) A model structure on $\mathcal{C}$ is:
- Three full subcategories of $Mor(\mathcal{C})$: $\mathcal{W}$, $\mathcal{C}of$, $Fib$
- Two functorial factorisations $(\alpha, \beta)$ and $(\alpha', \beta')$.

Such that the following holds:

① Call further
$\mathcal{W} \cap \mathcal{C}of$ "trivial cofibrations"
$\mathcal{W} \cap Fib$ "trivial fibrations"

Then for each $f: X \to Y$, we have
$\alpha(f)$ cofibr. $\alpha'(f)$ triv. cofibr.
$\beta(f)$ triv. fibr. $\beta'(f)$ fibr.

② 2-out-of-3: Consider

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \quad gf: X \to Z$$

Then if 2 out of $\{f, g, gf\}$ are weak equivalences, so is the $3^{rd}$.

③ $\mathcal{W}$, Fib, $\mathcal{C}of$ are closed under retracts.

④ Lifting
- Cofibr. have LLP wrt. triv. fibr
- Triv. Cofibr. have LLP wrt. fibr.

ALTERNATIVE AXIOMS:

③ (none)

④ $\mathcal{C}of$ are exactly those with LLP against $Fib \cap \mathcal{W}$; $Fib \cap \mathcal{W}$ are exactly those with RLP against $\mathcal{C}of$ } together with $(\alpha, \beta)$, this says that $(\mathcal{C}of, Fib \cap \mathcal{W})$ is a weak factorisation system.

$\mathcal{C}of \cap \mathcal{W}$ are exactly those with LLP against Fib; Fib are exactly those with RLP against $\mathcal{C}of \cap \mathcal{W}$ } $(\mathcal{C}of \cap \mathcal{W}, Fib)$ w.f.s.

A model category is a (co-)complete category with a model structure.

**Rmk:** This definition is self-dual: If $\mathcal{C}$ model cat, then $\mathcal{C}^{op}$ is model cat with

$\mathcal{W}(\mathcal{C}^{op}) = \mathcal{W}(\mathcal{C})^{op}$
$\mathcal{C}of(\mathcal{C}^{op}) = Fib(\mathcal{C})^{op}$
$Fib(\mathcal{C}^{op}) = \mathcal{C}of(\mathcal{C})^{op}$

**Lem:** (Retract argument): Assume

$$X \xrightarrow{i} Y \xrightarrow{p} Z, \quad f = pi$$

THEN: $f$ LLP wrt. $p$ ⟹ $f$ retract of $i$
(RLP wrt. $i$ ⟹ retract of $p$)

Pf: We have a lift

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f\downarrow & \nearrow^{r} & \downarrow p \\ Z & = & Z \end{array}$$

Thus, we get

$$\begin{array}{ccccc} X & = & X & = & X \\ f\downarrow & & i\downarrow & & \downarrow f \\ Z & \xrightarrow{r} & Y & \xrightarrow{p} & Z \end{array}$$

□

**Cor:** (Axiom 4') Let $\mathcal{C}$ model cat. Then
$f$ cofib. ⟺ $f$ LLP wrt. triv. fibr.
($f$ fibr. ⟺ $f$ RLP wrt. triv. cofib.)

Pf: Only show "⟸" of black version.
Can write $f = pi$ with
- $i$ cofib,
- $p$ trivial fibration.

Then $f$ LLP wrt. $p$
⟹ $f$ retract of $i$
⟹ $f$ cofibration. □

**Cor:** Let $\mathcal{C}$ model cat.
① Iso $\subseteq \mathcal{W} \cap \mathcal{C}of \cap Fib$
② Consider

$$\begin{array}{ccc} X & \to & X' \\ f\downarrow & & \downarrow f' \\ Y & \to & Y' \end{array}$$

Then we have
$f$ cofib ⟹ $f'$ cofib
($f$ fibr ⟸ $f'$ fibr.)

③ $\mathcal{C}of$ and Fib are closed under composition.

Pf: ① Isos have all LLP/RLP
② Let $p: E \to B$ triv. fibration. Then

$$\begin{array}{ccccc} X & \to & X' & \to & E \\ f\downarrow & & f'\downarrow & & \downarrow p \\ Y & \to & Y' & \to & B \end{array}$$

- since $f$ cofib
- since $Y'$ pushout

③ Let $p: E \to B$ triv. fibration. Then

$$\begin{array}{ccc} X & \to & E \\ f\downarrow & & \\ Y & & \downarrow p \\ g\downarrow & & \\ Z & \to & B \end{array}$$

- composition
- since $f$ cofib
- since $g$ cofib

□

**Def:** Let $\mathcal{C}$ model cat.
- $X$ cofibrant: $\emptyset \to X$ cofibr
- $X$ fibrant: $X \to *$ fibr
- $X$ bifibrant: both

**Constr:** Let $\mathcal{C}_{c/f/cf}$ be the full subcats of cofibr/fibr/bifibr. objects.
- Have inherited notions of weak equivalence on them.
- Have inclusions which preserve weak equivalences:

$$\begin{array}{ccc} \mathcal{C}_{cf} & \xrightarrow{i_f} & \mathcal{C}_f \\ i_c\downarrow & & \downarrow i_f \\ \mathcal{C}_c & \xrightarrow{j_c} & \mathcal{C} \end{array}$$

**Constr:** Have functorial factorisation

$$\emptyset \to QX \xrightarrow{\beta_X, \simeq} X \qquad (X \xrightarrow{\alpha_X, \simeq} RX \to *)$$

Thus, we get a (co)fibrant replacement
$Q: \mathcal{C} \to \mathcal{C}_c$
$R: \mathcal{C} \to \mathcal{C}_f$

together with a nat. eq. (i.e. each $\beta_X$ is a weak eq.)
$\beta: j_c \circ Q \Rightarrow 1_{\mathcal{C}}$ ($\alpha: 1_{\mathcal{C}} \Rightarrow j_f \circ R$)
resp. $\beta: Q \circ j_c \Rightarrow 1_{\mathcal{C}_c}$ ($\alpha: 1_{\mathcal{C}_f} \Rightarrow R \circ j_f$)

**Constr:** Have restricted functors
$R: \mathcal{C}_c \to \mathcal{C}_{cf}$
$Q: \mathcal{C}_f \to \mathcal{C}_{cf}$

Pf: $\emptyset \to X \xrightarrow{\alpha_X} RX$ cofibration. (both cofibrations)

## 3 The homotopy category

**Plan:** Use the developed theory to give an easier description of $Ho(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$

**Prop:** We have equivalences of categories:

$$\begin{array}{ccc} Ho(\mathcal{C}_{cf}) & \rightleftarrows & Ho(\mathcal{C}_f) \\ \downarrow\uparrow & & \uparrow\downarrow \\ Ho(\mathcal{C}_c) & \rightleftarrows & Ho(\mathcal{C}) \end{array}$$

(arrows $Ho(i_f)$, $Ho(Q)$, $Ho(i_c)$, $Ho(R)$, $Ho(j_c)$, $Ho(j_f)$)

Pf: Q and R preserve weak eq.:
Let $w: X \to Y$ weak eq.

$$\begin{array}{ccc} QX & \xrightarrow{\beta_X, \simeq} & X \\ Qw\downarrow & & \downarrow w, \simeq \\ QY & \xrightarrow{\beta_Y, \simeq} & Y \end{array}$$

By 2-out-of-3, also $Qw$ eq.
⟹ $Ho(Q)$ and $Ho(R)$ exist.
Finally, natural equivalences become nat. isos in $Ho(\mathcal{C})$. □

**Idea:** Give a more explicit definition of $Ho(\mathcal{C}_{cf})$.

**Def:** $\mathcal{C}$ model category.
- For $X \in \mathcal{C}$, a cylinder is $X \sqcup X \xrightarrow{i_0 \sqcup i_1} IX \xrightarrow{s, \simeq} X$ (composite $\nabla$), such that
  - $i_0 \sqcup i_1: X \sqcup X \to IX$ cofib
  - $s: IX \to X$ weak equiv.
  (always exists!)
- For $Y \in \mathcal{C}$, a path object is $Y \xrightarrow{c, \simeq} WY \xrightarrow{p_0 \times p_1} Y \times Y$ (composite $\Delta$), such that:
  - $p_0 \times p_1: WY \to Y \times Y$ fibr.
  - $c: Y \to WY$ weak equiv.
  (always exists!)
- For $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, a left homotopy is $H: IX \to Y$ with $H \circ (i_0 \sqcup i_1) = f \sqcup g$. Write $f \overset{l}{\sim} g$ iff $\exists H$.
- For $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, a right homotopy is $K: X \to WY$ with $(p_0 \times p_1) \circ K = f \times g$. Write $f \overset{r}{\sim} g$ iff $\exists K$.

**Lem:** Let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ and $Y \xrightarrow{h} Z$. Then
$f \overset{l}{\sim} g \Rightarrow hf \overset{l}{\sim} hg$ ($Z \to X$: $fh \overset{r}{\sim} gh$)

Pf: $h \circ H: IX \to Z$ ($K \circ h: Z \to WY$) □

**Lem:** If $X$ cofibrant ($Y$ fibrant), then on $\mathcal{C}(X, Y)$:
① $\overset{l}{\sim}$ ($\overset{r}{\sim}$) is an equivalence relation.
② $\overset{l}{\sim}$ implies $\overset{r}{\sim}$.
(Hovey, Prop. 1.2.5)

**Constr:** If $X$ cofibrant and $Y$ fibrant, we have a unique hty. relation and we set

$$[X, Y] := \mathcal{C}(X, Y)/\sim.$$

**Constr:** On $\mathcal{C}_{cf}$, being homotopic is compatible with composition, i.e.

$$\begin{array}{ccc} \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) & \xrightarrow{\circ} & \mathcal{C}(X,Z) \\ \downarrow & & \downarrow \\ [Y,Z] \times [X,Y] & \dashrightarrow & [X,Z] \end{array}$$

Thus, we obtain a quotient category

$$\delta: \mathcal{C}_{cf} \to \mathcal{C}_{cf}/\sim$$

**Def:** $f: X \to Y$ in $\mathcal{C}_{cf}$ is called hty eq if
$\exists g: Y \to X$: $gf \sim id_X$ AND $fg \sim id_Y$

**Thm:** (Whitehead) For maps among bifibrant objects:
homotopy eq. ⟷ weak equivalence.
(Hovey, Prop. 1.2.8)

**Prop:** $\delta: \mathcal{C}_{cf} \to \mathcal{C}_{cf}/\sim$ has the u.P. of $Ho(\mathcal{C}_{cf})$.

Pf:
- Clearly $\delta(f)$ iso for $f$ hty eq (= weak eq.)
- Let $F: \mathcal{C}_{cf} \to \mathcal{D}$ which send weak eq. to isos. and let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be homotopic. Then

$$X \sqcup X \xrightarrow{i_0 \sqcup i_1} IX \xrightarrow{H} Y \quad (f \sqcup g) \implies f = Hi_0,\ g = Hi_1$$

However, recall that

$$X \sqcup X \xrightarrow{i_0 \sqcup i_1} IX \xrightarrow{s, \simeq} X \quad (\nabla) \implies si_0 = id_X = si_1 \text{ and } s \text{ weak eq.}$$

Side remark: If $X$ bifibrant, so is $IX$:
- Have fibrations $IX \xrightarrow{\simeq} X \to *$
- Also $X \sqcup X$ is cofibrant: (pushout of $\emptyset \to X$)
Thus, we have cofibrations $\emptyset \to X \sqcup X \xrightarrow{i_0 \sqcup i_1} IX$

Thus, $Fs$ iso and

$Fg = FH \circ Fi_0$
$= FH \circ (Fs)^{-1} \circ F(si_0)$
$= FH \circ (Fs)^{-1} \circ F(si_1)$
$= Ff$.

⟹ ∃ extension

$$\begin{array}{ccc} \mathcal{C}_{cf} & \xrightarrow{F} & \mathcal{D} \\ \delta\downarrow & \nearrow_{\tilde{F}} & \\ \mathcal{C}_{cf}/\sim & & \end{array}$$

- Since $\delta$ is
  - bijective on objects,
  - full,
  
  $\tilde{F}$ is uniquely determined. □

**Cor:** (Fundamental theorem for model categories):
① We have an equivalence of categories

$$Ho(\mathcal{C}) \xrightarrow[Ho(QR) \cong Ho(RQ)]{\simeq} Ho(\mathcal{C}_{cf}) = \mathcal{C}_{cf}/\sim$$

② We have natural bijections

$$[QRX, QRY] \cong Ho(\mathcal{C})(X,Y) \cong [RQX, RQY]$$

(in particular, $Ho(\mathcal{C})$ is locally small!)

③ Even better:
$Ho(\mathcal{C})(X,Y) \cong [QX, RY]$ } (Hovey, Prop. 1.2.5)

## 4 Quillen adjunctions / equivalences

**Idea:** The "correct" notion of morphism between model categories should induce morphisms between their homotopy categories.

⇝ Clearly $F(\mathcal{W}) \subseteq \mathcal{W}'$ would suffice, but often, this is too much to ask for.

⇝ We can do better than that by demanding less.

**Def:** Let $\mathcal{C}, \mathcal{D}$ model cats. A functor $F: \mathcal{C} \to \mathcal{D}$ is called left (right) Quillen if:
- F is left (right) adjoint
- F preserves (trivial) cofibrations.

**Prop:** Given an adjunction $F: \mathcal{C} \leftrightarrows \mathcal{D}: U$, the following are equivalent:
① F left Quillen,
② U right Quillen.

Then call $(F, U)$ Quillen pair.
(nice exercise!)

**Lem:** (Ken Brown) Let $F: \mathcal{C} \to \mathcal{D}$ be left (right) Quillen and $f: X \to Y$ weak eq. between (co)fibrant obj.
⟹ $Ff$ weak equivalence
(Hovey, Lem. 1.1.12)

actually, Ken Brown's lemma says even more, but we need only this

**Constr:** Let $\mathcal{C}$ and $\mathcal{D}$ be model cats.
① If $F: \mathcal{C} \to \mathcal{D}$ is left Quillen, define the total left derived functor

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{Q} & \mathcal{C}_c & \xrightarrow{F} & \mathcal{D} \\ \downarrow & & \downarrow & & \downarrow \\ LF: Ho(\mathcal{C}) & \dashrightarrow & Ho(\mathcal{C}_c) & \dashrightarrow & Ho(\mathcal{D}) \end{array}$$

② If $U: \mathcal{D} \to \mathcal{C}$ is right Quillen, define the total right derived functor

$$\begin{array}{ccccc} \mathcal{D} & \xrightarrow{R} & \mathcal{D}_f & \xrightarrow{U} & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ RU: Ho(\mathcal{D}) & \dashrightarrow & Ho(\mathcal{D}_f) & \dashrightarrow & Ho(\mathcal{C}) \end{array}$$

**Ex:** Let R comm ring, M R-mod. Consider

$$F = M \otimes_R -: Ch^R \to Ch^R$$

(proj. m.s. → inj. m.s.)

- In general, F does not preserve quasiisos:

$$\begin{array}{ccccccccccc} C_\bullet = & \cdots \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\ f\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ D_\bullet = & \cdots \to & 0 & \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \to & \mathbb{Z}/2 & \to & 0 \end{array}$$

(degree [0] at the $\mathbb{Z}/2$ position)

Then $H_*(f)$ iso ($H_*C = H_*D = 0$), but $\mathbb{Z}/2 \otimes f$ is not:

$$H_2(\mathbb{Z}/2 \otimes D) = \mathbb{Z}/2 \neq 0 = H_2(\mathbb{Z}/2 \otimes C)$$

- However, F is left Quillen:
  - F is left adjoint to $Hom_R(M, -)$,
  - Let $f: C \to D$ be a cofibration wrt. the proj. m.s.
    ⟹ $f$ levelwise split monic
    ⟹ $M \otimes f$ monic
    ⟹ $M \otimes f$ cofibration
    } F preserves cofibr.
  - If $f$ additionally quasiiso, then also $M \otimes f$ quasiiso:
    - We have
    
    $$\begin{array}{ccc} C & \to & 0 \\ f\downarrow & & \downarrow i \\ D & \to & coker(f) \end{array} \implies i \text{ cofibration}$$
    
    - By LES in $H_*$, $coker(f)$ acyclic
      ⟹ $i$ trivial cofibration
      ⟹ $i$ LLP wrt. fibrations (= epis)
      ⟹ $coker(f)$ projective object in $Ch^R$
      ⟹ $coker(f)$ split exact
    - Since F preserves cokernels and chain homotopies: $coker(M \otimes f)$ acyclic
    
    By LES in $H_*$, we get that $M \otimes f$ quasiiso
    } F preserves trivial cofibr.

- Can consider its left derived functor
  $LF: Ho(Ch) \to Ho(Ch)$
  
  Since this was constructed by cofibrant replacement:
  
  $$R\text{-Mod} \to Ho(Ch) \xrightarrow{LF} Ho(Ch) \xrightarrow{H_n} R\text{-Mod}$$
  $$N \mapsto N[0] \mapsto M \otimes PN \mapsto H_n(M \otimes QN)$$
  
  ($PN$: projective resolution)
  
  is just $Tor_n(M, -) = L_nF$.

- Note that $PN$ is cofibrant and $PN \to N$ is a quasiiso
- Since $\beta_N: QN \to N$ is a trivial fibration, we have

$$\begin{array}{ccc} 0 & \to & QN \\ \downarrow & \nearrow^{\varphi} & \downarrow \beta_N \\ PN & \xrightarrow{\simeq} & N \end{array}$$

- By 2-out-of-3, $\varphi$ equivalence between cofibrant objects
  ⟹ $M \otimes \varphi: M \otimes PN \to M \otimes QN$ is an equivalence

**Lem:** Let $F: \mathcal{C} \leftrightarrows \mathcal{D}: U$ be a Quillen pair.
⟹ Have derived adjunction
$LF: Ho(\mathcal{C}) \leftrightarrows Ho(\mathcal{D}): RU$
(Hovey, Lem 1.3.10)

**Def:** A Quillen pair $F: \mathcal{C} \leftrightarrows \mathcal{D}: U$ is called Quillen equivalence, if the derived adjunction

$$LF: Ho(\mathcal{C}) \rightleftarrows Ho(\mathcal{D}): RU$$

is an equivalence of categories.

**Ex/Outlook:** Wrt. the model structures mentioned above, we will see that the adjunction

$$|\cdot|: sSet \leftrightarrows Top: Sing$$

is a Quillen equivalence.

⟹ "Top and sSet have the same homotopy theory"