① Loop spaces & H-spaces

Recall: The easiest definition of "loop space" is ... Consider e.g. for a based space X its loop space

$\Omega X = \{\text{based maps } S^1 \to X\}$

We have multiplication by concatenation

$\Omega X \times \Omega X \longrightarrow \Omega X$

This is not strictly associative/unital, but only "up to homotopy".

Def: An H-space is a based space (M,e) together with a based multiplication

$\mu: M \times M \to M$, $(e,e) \mapsto e$

which is H-unital, i.e. there are based homotopies.

We define special properties: ① H-associativity ② H-commutativity

Ex: ① Loop spaces with concatenation.
② Labelled configuration spaces $C(\mathbb{R}^n; X)$
③ Moduli spaces of surfaces $\coprod_g \mathcal{M}_g$

Rmk: H-spaces have a lot of structure:

① $H_*(M)$ has an algebra structure:
$H_*(M) \otimes H_*(M) \to H_*(M \times M) \to H_*(M)$ (Pontryagin product)
- associative if M H-ass.
- gr-comm if M H-comm.

② For each based X, the set $[X,M]_*$ is a unital magma. (true monoid if M H-ass.) In particular:
- $\pi_0(M) = [S^0, M]_*$ is a discrete monoid for M H-ass.
- $\pi_1(M) = [S^1, M]_*$ has two unital products which interchange

$\Longrightarrow$ Both agree and are commutative (Eckmann – Hilton)

$\Longrightarrow$ $\pi_1(M)$ abelian ⚠ This only affects the path component of e!

③ S^0, S^1, S^3, S^7 are the only spheres which carry an H-space structure. (Adams 1962)

Q1: Is each H-space $\simeq \Omega(\text{some } X)$?
NO! Need at least "$\pi_0 M = \pi_1 X$ group" (called group-like)

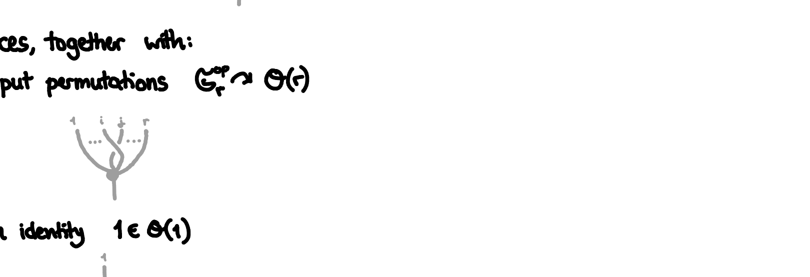
Q2: Is each group-like H-space $\simeq \Omega(\text{some } X)$?
NO! E.g. $S^7 \not\simeq \Omega X$
(would need $H^*(X) \cong \mathbb{Z}[\alpha]$, $|\alpha| = 8$ which is not possible, Steenrod 1962)

⇝ Need to get better control over the possibilities of multiplying r elements

Observation: The n-fold loop space

$\Omega^n X \cong \{ f: I^n \to X ;\ f(\partial I) = * \}$

has an entire space parametrising r-fold multiplications:



space of them: $\mathcal{C}_2(3)$

Q3: Is every group-like "space with multiplications parametrised by $\mathcal{C}_n$" an $\Omega^n(\text{some } X)$?
YES! (Recognition principle, May 1972)

2 Little n-cubes (n<∞)

Def: An operad is a sequence $\mathcal{O}(0), \mathcal{O}(1), \dots$ of spaces, together with:
- input permutations $\mathfrak{S}_r^{op} \curvearrowright \mathcal{O}(r)$
- an identity $1 \in \mathcal{O}(1)$
- compositions $\mathcal{O}(r) \times \prod_{i=1}^{r} \mathcal{O}(s_i) \longrightarrow \mathcal{O}(s_1 + \dots + s_r)$, $(c; d_1, \dots, d_r) \longmapsto c(d_1, \dots, d_r)$

sth. "the obvious" holds.
We additionally demand $\mathcal{O}(0) = \{0\}$

Def: An $\mathcal{O}$-alg. is a space M together with

$\chi^r: \mathcal{O}(r) \times M^r \longrightarrow M$

sth. "the obvious" holds. Then M is canonically based by

$\chi^0: \{0\} = \mathcal{O}(0) \times M^0 \longrightarrow M$, $0 \longmapsto e$.

Rmk: If $\mathcal{O}(2) \neq \emptyset$, fix $c \in \mathcal{O}(2)$. Now for each $\mathcal{O}$-alg M, consider

$\mu: M \times M \longrightarrow M$
$(x, x') \longmapsto \chi^2(c; x, x')$.

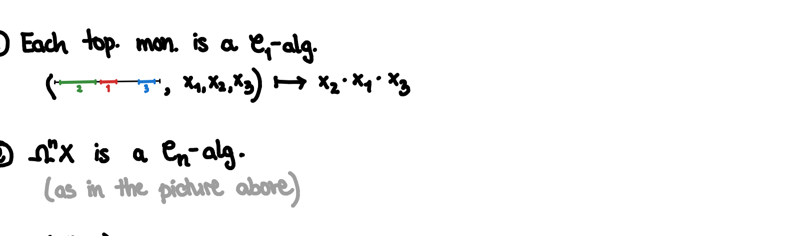
(This is a based map as $\chi^2(c; e, e) = \chi^2(c(0,0)) = \chi^0(0) = e$.)

(C1) If $c(1,0) \sim 1 \sim c(0,1)$ (in the same path comp.), then M H-unital,
(C2) If $c(c,1) \sim c(1,c)$, then M H-associative,
(C3) If $(12)^* c \sim c$, then M H-commutative.

Constr: (Little n-cubes) For $n \geq 1$, the spaces

$\mathcal{C}_n(r) := \{ c: \coprod^r I^n \hookrightarrow I^n \text{ rectilinear} \}$

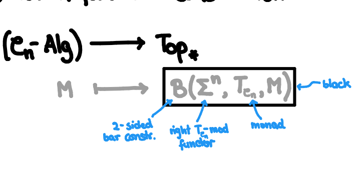
form an operad, with composition



Rmk: ① Each $c \in \mathcal{C}_n(2)$ satisfies (C1) + (C2)
$\Rightarrow$ $\mathcal{C}_n$-alg. are H-ass. H-spaces.

② For $n \geq 2$, $\mathcal{C}_n(2)$ is connected ($\Rightarrow$ (C3))
$\Rightarrow$ For $n \geq 2$, $\mathcal{C}_n$-alg are also H-comm.

③ There are inclusions $\mathcal{C}_n \hookrightarrow \mathcal{C}_{n+1}$

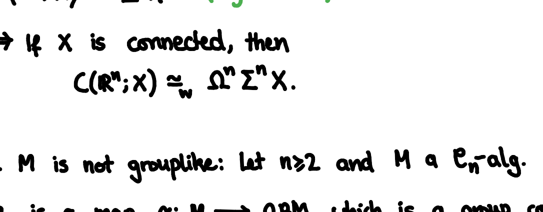


so each $\mathcal{C}_{n+1}$-alg. is a $\mathcal{C}_n$-alg.

Expl: ① Each top. mon. is a $\mathcal{C}_1$-alg.
$(\text{—}_2\text{—}_1\text{—}_3 , x_1, x_2, x_3) \longmapsto x_2 \cdot x_1 \cdot x_3$

② $\Omega^n X$ is a $\mathcal{C}_n$-alg.
(as in the picture above)

③ $C(\mathbb{R}^n; X)$ is a $\mathcal{C}_n$-alg.



Constr: There is an 'n-fold bar construction'

$B^n: (\mathcal{C}_n\text{-Alg}) \longrightarrow \mathrm{Top}_*$
$M \longmapsto \boxed{B(\Sigma^n, T_{\mathcal{C}_n}, M)}$ ← black box

2-sided bar constr. / right $T_{\mathcal{C}_n}$-mod functor / monad

> **Thm:** (Recognition principle, May 1972)
> If M group-like $\mathcal{C}_n$-alg, then
> $M \simeq_w \Omega^n B^n M$
> (actually, there is a zig-zag of equivalences of $\mathcal{C}_n$-algebras)

Expl: $B^n C(\mathbb{R}^n; X) \simeq \Sigma^n X$ (Segal 1973)
$\Rightarrow$ If X is connected, then
$C(\mathbb{R}^n; X) \simeq_w \Omega^n \Sigma^n X$.

Constr: If M is not grouplike: Let $n \geq 2$ and M a $\mathcal{C}_n$-alg.
- There is a map $\gamma: M \longrightarrow \Omega BM$ which is a group completion, i.e.

$$\begin{tikzcd} H_*(M) \arrow[r, "\gamma_*"] \arrow[d] & H_*(\Omega BM) \\ H_*(M)[\pi_0^{-1}] \arrow[ur, dashed, "\cong"'] & \end{tikzcd}$$

Pontrjagin algebra is graded commutative

← i.e. if M grouplike, then γ weak eq.

(Group completion thm, McDuff/Segal 1976)

- One then shows $\Omega BM \simeq_w \Omega^n B^n M$.
- If $\pi_0 M \cong \mathbb{N}$, choose $s \in M_1$. Then

$\mathrm{hocolim}(M \xrightarrow{\cdot s} M \xrightarrow{\cdot s} \dots) \simeq \mathbb{Z} \times M_\infty$

$M \xrightarrow{\cdot s} M \xrightarrow{\cdot s} M$; M_0, M_1, M_2, M_3, M_4; M_∞

Then we can show:
- $\tilde{\gamma}: \mathbb{Z} \times M_\infty \longrightarrow \Omega BM$ is a H_*-eq.,
- For $\pi := \pi_1(M_\infty)$, $[\pi, \pi] \leq \pi$ perfect

From this, we get

$\Omega BM \simeq_w \mathbb{Z} \times M_\infty^+$

Quillen's plus construction:
Let X space, $N \trianglelefteq \pi_1 X$ perfect.
Then ∃ rel. CW cpx $j: X \hookrightarrow X^+$ sth.
- j H_*-equiv.,
- $j_*: \pi_1 X \longrightarrow \pi_1 X^+$ is $/N$

Ex: Consider $M = C(\mathbb{R}^2; S^0) \simeq \coprod_{r \geq 0} B\beta_r$ (braid group)

$\Rightarrow (\mathbb{Z} \times) B\beta_\infty^+ \simeq \Omega^2 B^2 C(\mathbb{R}^2; S^0) \simeq \Omega^2 S^2_0$

This can now be combined with a homological stability result

$H_i(\beta_k) \cong H_i(\beta_\infty)$ for $\frac{i}{k}$ small
$\cong H_i(B\beta_\infty^+)$
$\cong H_i(\Omega^2 S^2_0)$

Fairy tale: Being a $\mathcal{C}_1$-algebra is much better than just being H-associative:
Once having chosen contractions of $\mathcal{C}_1(r)$, we have consistent choices of homotopies:

- $a(bc)$ —— $(ab)c$ $(A_3$, H-ass.$)$ $\times M^3 \longrightarrow M$
- $(ab)(cd)$, $a(b(cd))$, $((ab)c)d$, $a((bc)d)$ —— $(a(bc))d$ (A_4) $\times M^4 \longrightarrow M$ (cf. pentagon law for monoidal categories)

⋮ ⋮ $\times M^5 \longrightarrow M$

(A_∞)

forms an operad equivalent to $\mathcal{C}_1$ (Stasheff associahedron)

H-spaces with a consistent choice of these maps are called A_∞-spaces. One shows

$(A_\infty\text{-spaces}) \simeq (\mathcal{C}_1\text{-algebras})$

There are different operads modelling A_∞-spaces, called A_∞-opd.

3 Infinite loop spaces and spectra

Idea: Recall $\mathcal{C}_1 \hookrightarrow \mathcal{C}_2 \hookrightarrow \cdots$
What if M is an algebra over all $\mathcal{C}_n$ compatibly?

$\mathcal{C}_n(r) \times M^r \longrightarrow \mathcal{C}_{n+1}(r) \times M^r$, $\searrow M \swarrow$

Ex: ① Each abelian top. monoid, by
$\mathcal{C}_n(r) \times M^r \longrightarrow M$
$(c; x_1, \dots, x_r) \longmapsto x_1 + \dots + x_r$

② $C(\mathbb{R}^\infty; X)$

We define the operad $\mathcal{C}_\infty = \varinjlim \mathcal{C}_n$ and want to state

> If M is a grouplike $\mathcal{C}_\infty$-algebra, then $M \simeq_w \Omega^\infty B^\infty M$

More formally, this should mean that there is

$\left(\begin{array}{l} M \simeq_w B^0 M, B^1 M, \dots \\ \text{together with } B^n M \xrightarrow{\sim} \Omega B^{n+1} M \end{array} \right) =: B^\infty M$ "Ω-spectrum"

Rmd: (Spectra)
- A spectrum is a seq. $(E_n)_{n \geq 0}$ of based spaces, together with maps $\sigma_n: \Sigma E_n \longrightarrow E_{n+1}$
- (E, σ) Ω-sp. if $\bar{\sigma}_n: E_n \longrightarrow \Omega E_{n+1}$ is a weak equiv.
Have a functor turning each sp. into an Ω-sp.

$Q: \mathrm{Sp} \longrightarrow \mathrm{Sp}$
$E \longmapsto (\varinjlim_i \Omega^i E_{n+i})_{n \geq 0}$ "fibrant replacement"

- For a based space, we have the suspension spectrum with $(\Sigma^\infty X)_n = \Sigma^n X$. Obtain (Quillen) adjunction

$\Sigma^\infty: \mathrm{Top}_* \rightleftarrows \mathrm{Sp} : (-)_0$

- We have a derived version of $(-)_0$ by

$\Omega^\infty: \mathrm{Sp} \longrightarrow \mathrm{Top}_*$
$E \longmapsto (QE)_0$

Ω^∞ is homotopically nice:
- $E \simeq E' \Longrightarrow \Omega^\infty E \simeq \Omega^\infty E'$,
- If E Ω-sp, then $\Omega^\infty E \simeq E_0$.

Constr: We have functors

$B^0, B^1, B^2, \dots : (\mathcal{C}_\infty\text{-alg}) \longrightarrow \mathrm{Top}_*$

by $B^n M = \varinjlim_i \Omega^i B(\Sigma^{n+i}, T_{\mathcal{C}_{n+i}}, M)$, and there are natural identifications $B^n M \xrightarrow{\simeq} \Omega B^{n+1} M$

$\Longrightarrow$ We get a functor
$B^\infty: (\mathcal{C}_\infty\text{-alg}) \longrightarrow \mathrm{Sp}$

Ex: ① $B^\infty C(\mathbb{R}^\infty; X) \simeq \Sigma^\infty X$ (Segal 1973)
② If A discrete ab. gp, then $B^\infty A \simeq HA$ (Eilenberg – MacLane sp.)

> **Thm:** If M is a $\mathcal{C}_\infty$-algebra, then
> $\Omega BM \simeq_w \Omega^\infty B^\infty M$.

(i.e. if M grouplike, then $M \simeq_w \Omega^\infty B^\infty M$)

Rmk: (Connective spectra)
- For a spectrum E and $k \in \mathbb{Z}$, we let
$\pi_k(E) = \varinjlim_i \pi_{k+i}(E_i)$ stable homotopy groups
- Call a spectrum connective if $\pi_k(E) = 0$ for $k < 0$. Then
$\pi_k(E) = \pi_k(\Omega^\infty E)$
- The above machinery actually gives a Quillen equivalence

$B^\infty: \begin{pmatrix} \text{grouplike} \\ \mathcal{C}_\infty\text{-algebras} \end{pmatrix} \rightleftarrows \begin{pmatrix} \text{connective} \\ \text{spectra} \end{pmatrix} : \Omega^\infty$

Rmk: One can show that

$B^\infty: (\mathrm{TopAb}, \otimes) \longrightarrow (\mathrm{Sp}, \wedge)$

is lax monoidal, and each top. ab gp is a $\mathbb{Z}$-module $\quad \mathbb{Z} \otimes A \to A$, $H\mathbb{Z} \wedge B^\infty A \to B^\infty A$

$\Rightarrow$ For a top. ab. gp. A,
$B^\infty A$ is a connective $H\mathbb{Z}$-module.

⌜And this recovers the Dold – Kan correspondence:

$$\begin{tikzcd} \mathrm{TopAb} \arrow[r, "B^\infty"] \arrow[d, "\mathrm{Sing}"'] & \text{conn. } H\mathbb{Z}\text{-modules} \arrow[d, leftrightarrow, "\simeq"] \\ s\mathrm{Ab} \arrow[r, shift left, "\mathrm{D.K.}"] & \mathrm{Ch}_{\geq 0} \arrow[l, shift left] \end{tikzcd}$$ ⌟

Ex: ① If X is connected, then
$C(\mathbb{R}^\infty; X) \simeq_w \Omega^\infty \Sigma^\infty X$.

② Consider $M = C(\mathbb{R}^\infty; S^0) = \coprod_{r \geq 0} B\mathfrak{S}_r$. Then

$\mathbb{Z} \times B\mathfrak{S}_\infty^+ \simeq \Omega^\infty B^\infty C(\mathbb{R}^\infty; S^0) \simeq \Omega^\infty S^\infty$ (Barratt – Priddy – Quillen) $\quad \pi_k \Omega^\infty S^\infty \cong \pi_k^s$

Fairy tale 2/ Intermezzo: As for the A_∞-case, there are other operads which encode "∞-coherent commutativity" and have "B^∞".
An E_∞-operad is an operad $\mathcal{O}$ sth:
- $\mathcal{O}(r)$ is $\mathfrak{S}_r$-free $\forall r \geq 0$,
- $\mathcal{O}(r) \simeq *$ (non-eqvly) $\forall r \geq 0$.

Examples:
① $\mathcal{C}_\infty$
② The linear isometries operad $\mathcal{L}$:
Consider $\mathbb{R}^\infty$ with standard inner product and set
$\mathcal{L}(r) := \{ \text{isometries } \varphi: \bigoplus^r \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty \}$

③ The Barratt-Eccles operad $\mathcal{E}$:

$\mathcal{E}(r) := B(\mathfrak{S}_r^{\boxtimes})$

category with
- obj = $\mathfrak{S}_r$
- exactly one morph $\sigma \to \tau$

With composition by 'cabelling'

Can show: For each sym. mon. category C,
$BC \simeq$ an $\mathcal{E}$-Alg. (as H-spaces)

③ Let R be a ring
$\Rightarrow$ Have sym. mon. cat $GL(R)$ with
- obj = 0, 1, 2, ...
- morphisms $n \to n$ are $GL_n(R)$, no others
- Monoidal sum = block sums $\begin{pmatrix} * & \\ & * \end{pmatrix}$

$\Rightarrow$ Have $\mathcal{E}$-alg $\simeq \coprod_{n \geq 0} BGL_n(R)$
$\Rightarrow$ Its group-completion

$\Omega B\left(\coprod_{n \geq 0} BGL_n(R)\right) = \mathbb{Z} \times \underbrace{BGL_\infty(R)^+}_{\text{used by Quillen to define the algebraic K-theory of R.}}$

is the 0^{th} space of an Ω-spectrum.