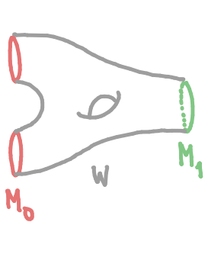
THE HOMOTOPY TYPE OF THE COBORDISM CATEGORY

(Galatius – Madsen – Tillmann – Weiss 2009)

1 Warm-up: "Discrete" cobordism categories

Axiom: Every manifold today is smooth and compact.

Rmd: For two closed $(d-1)$-folds M_0 and M_1, a cobordism between them is a d-fold W with boundary $\partial W = \partial_0 W \sqcup \partial_1 W$ sth $\partial_k W \cong M_k$

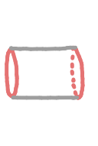


Constr: Turn this into a category $h\mathcal{C}ob_d$:

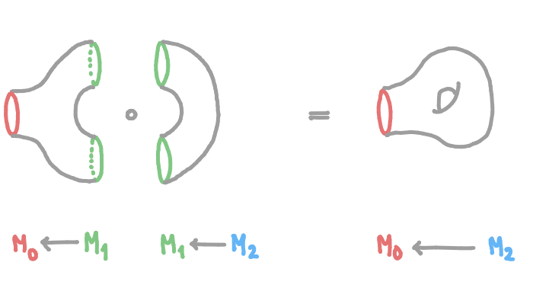
① objects: closed $(d-1)$-folds M

② morphisms $M_0 \leftarrow M_1$: isomorphism classes of d-dim cobordisms $(W, \partial_0 W, \partial_1 W)$ together with identifications $M_k \cong \partial_k W$ (isomorphisms have to preserve these identifications)

③ identities: $id_M = I \times M$

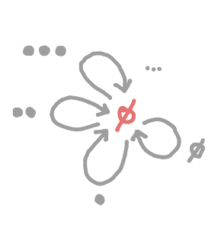


④ composition: Gluing



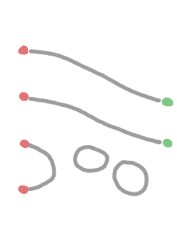
Expl:

- Dimension 0: $h\mathcal{C}ob_0$



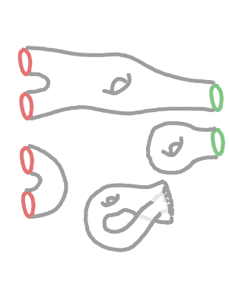
$\cong \mathbb{N}$

- Dimension 1: $h\mathcal{C}ob_1$
  - isomorphism classes of objects: $\emptyset, \bullet, \bullet\bullet, \bullet\bullet\bullet, \ldots$
  - morphisms look like this:



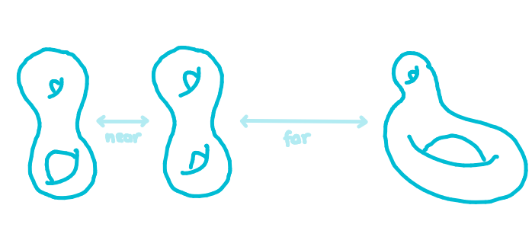
(e.g. $h\mathcal{C}ob_1(\emptyset \leftarrow \emptyset) = \{\emptyset, S^1, \amalg^2 S^1, \ldots\} \cong \mathbb{N}$)

- Dimension 2: $h\mathcal{C}ob_2$
  - isomorphism classes of objects: $\emptyset, S^1, \amalg^2 S^1, \ldots$
  - morphisms look like this:



2 Topological cobordism categories

Motivation: There is not only the set of isomorphism classes of manifolds (resp. cobordisms), but also, for each isomorphism class $[M]$, the space of "how M may look":



One way to formalise this: Consider

$$\mathrm{Mod}(M) = \left\{ \begin{smallmatrix} \text{submanifolds of } \mathbb{R}^\infty \\ \text{of type } [M] \end{smallmatrix} \right\} = \frac{\mathrm{Emb}(M, \mathbb{R}^\infty)}{\mathrm{Diff}(M)} \simeq B\mathrm{Diff}(M)$$

This is in line with the usual transition from classification to moduli problems

	classification invariant	moduli spaces	
finite-dim. $\mathbb{R}$-vsp	dimension	$\mathrm{Mod}(\mathbb{R}^n) = \frac{\mathrm{Emb}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^\infty)}{GL_n(\mathbb{R})} = Gr_n(\mathbb{R}^\infty) \simeq BO(n)$	Graßmannian
0-dim mfds	#points	$\mathrm{Mod}(\underline{n}) = \frac{\mathrm{Emb}(\underline{n}, \mathbb{R}^\infty)}{\mathfrak{S}_n} = C_n(\mathbb{R}^\infty) \simeq B\mathfrak{S}_n$	unordered conf. space
closed 1-dim mfds	#circles	$\mathrm{Mod}(\amalg^n S^1) = \frac{\mathrm{Emb}(\amalg^n S^1, \mathbb{R}^\infty)}{\mathrm{Diff}(\amalg^n S^1)}$	
connected, oriented surfaces with one boundary curve	genus	$\mathrm{Mod}(\Sigma_{g,1}) = \frac{\mathrm{Emb}_\partial(\Sigma_{g,1}, \mathbb{R}^\infty)}{\mathrm{Diff}^{SO}_\partial(\Sigma_{g,1})} \simeq \mathcal{M}_{g,1} \simeq B\mathrm{Diff}^{SO}_\partial(\Sigma_{g,1})$	moduli sp. of surfaces

Rmd: A topological category is given by:

① a space $ob(\mathcal{C})$ of objects
② a space $mor(\mathcal{C})$ of morphisms
③ three maps
$$mor(\mathcal{C}) \overset{d_1 \text{ "source"}}{\underset{d_0 \text{ "target"}}{\rightrightarrows}} ob(\mathcal{C}), \quad ob(\mathcal{C}) \xrightarrow{s_0} mor(\mathcal{C})$$
④ a composition map
$$mor(\mathcal{C}) \underset{ob\mathcal{C}}{\times} mor(\mathcal{C}) \xrightarrow{\text{"}\circ\text{"}} mor(\mathcal{C})$$

sth [...]

Constr: Let $d \geq 0$. Define the d-dim cobordism category $\mathcal{C}ob_d$:

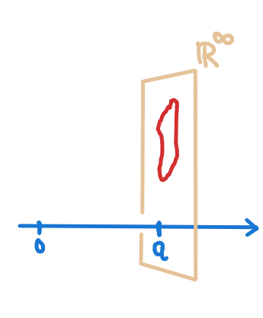
① Objects

For a closed $(d-1)$-fold M let

$$\mathrm{Emb}(M) := \left\{ (a, m) ;\ \begin{smallmatrix} a \in \mathbb{R} \\ m: M \hookrightarrow \{a\} \times \mathbb{R}^\infty \end{smallmatrix} \right\} \subseteq \mathbb{R} \times \mathrm{Emb}(M, \mathbb{R} \times \mathbb{R}^\infty)$$

Then $\mathrm{Diff}(M) \curvearrowright \mathrm{Emb}(M)$ by precomposition and we let

$$ob(\mathcal{C}ob_d) = \coprod_{[M]} \frac{\mathrm{Emb}(M)}{\mathrm{Diff}(M)}$$



② Morphisms

For a cobordism $(W, \partial_0 W, \partial_1 W)$ let

$$\mathrm{Diff}(W) := \{ \varphi : W \xrightarrow{\cong} W ;\ \varphi(\partial_k W) = \partial_k W \}$$

$$\mathrm{Emb}(W) := \left\{ (a_0, a_1, w) ;\ \begin{smallmatrix} a_0 < a_1 \ (\text{"}\leq\text{" if } W = \emptyset) \\ w: W \hookrightarrow [a_0, a_1] \times \mathbb{R}^\infty \\ w^{-1}(\{a_k\} \times \mathbb{R}^\infty) = \partial_k W \end{smallmatrix} \right\} \subseteq \mathbb{R}^2 \times \mathrm{Emb}(W, \mathbb{R} \times \mathbb{R}^\infty)$$

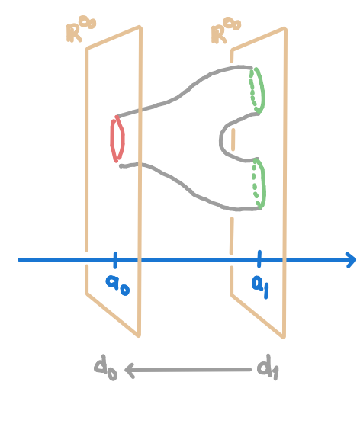
Then $\mathrm{Diff}(W) \curvearrowright \mathrm{Emb}(W)$ by precomposition and we let

$$mor(\mathcal{C}ob_d) = \coprod_{[W]} \frac{\mathrm{Emb}(W)}{\mathrm{Diff}(W)} \underset{\mathbb{R} \times \{a\}}{\cup} ob(\mathcal{C}ob_d)$$

identities are "cylinders of length 0"

③ Three maps

- $d_k[a_0, a_1, w] = [a_k, w|_{\partial_k W}]$ $\quad k = 0, 1$
- s_0 is just the inclusion $ob(\mathcal{C}ob_d) \hookrightarrow mor(\mathcal{C}ob_d)$



④ Composition

$$\overbrace{[a_0, a_1, w] \circ [a_0', a_1', w']}^{=} = [a_0, a_1', w \cup_\varphi w']$$

$$\begin{tikzcd} \partial_1 W \arrow[rr, "\exists!\varphi", "\cong"'] \arrow[dr, "w"'] & & \partial_0 W' \arrow[dl, "w'"] \\ & \{a_1\} \times \mathbb{R}^\infty & \end{tikzcd}$$

Expl:

- $\mathcal{C}ob_0 \simeq \coprod_{n \geq 0} C_n(\mathbb{R}^\infty) \simeq \coprod_{n \geq 0} B\mathfrak{S}_n$
- $\mathcal{C}ob_1(\emptyset \leftarrow \emptyset) \simeq \coprod_{n \geq 0} B\mathrm{Diff}(\amalg^n S^1)$
- dimension 2: wait until §4!

Rmk: $h\mathcal{C}ob_d$ is equivalent to the homotopy category of $\mathcal{C}ob_d$ ("make" ob discrete and take π_0 of $\mathcal{C}ob_d(y \leftarrow x)$)

3 The homotopy type of $\mathcal{C}ob_d$

Rmd: A topological category gives rise to a simplicial space $N\mathcal{C}$, the nerve:

$$(N\mathcal{C})_n := \overbrace{mor(\mathcal{C}) \underset{ob\mathcal{C}}{\times} \cdots \underset{ob\mathcal{C}}{\times} mor(\mathcal{C})}^{n \text{ times}}$$

$$d_i(f_n, \ldots, f_1) := \begin{cases} (f_n, \ldots, f_2) & i = 0, \\ (f_n, \ldots, f_{i+1} f_i, \ldots, f_1) & 0 < i < n, \\ (f_{n-1}, \ldots, f_1) & i = n. \end{cases}$$

$$s_i(f_n, \ldots, f_1) := (f_n, \ldots, f_{i+1}, id, f_i, \ldots, f_1)$$

We define the bar construction

$$B\mathcal{C} := |N\mathcal{C}| = \int^{n \in \Delta} (N\mathcal{C})_n \times \Delta^n$$

Expl: Let G be a topological group. Regard G as a top. category with

- a single object.
- morphisms $= G$.

Then BG is the classifying space of G (e.g. $B\mathbb{Z} \simeq S^1$)

Q: What can we say about $B\mathcal{C}ob_d$?

Expl: Recall $\mathcal{C}ob_0 \simeq \coprod_{n \geq 0} C_n^{\text{Moore}}(\mathbb{R}^\infty) \simeq \coprod_{n \geq 0} B\mathfrak{S}_n$

(and indeed, the monoid structure from the composition on $\mathcal{C}ob_0$ agrees with the Moore strictified E_∞-alg. structure on $\coprod_{n \geq 0} B\mathfrak{S}_n$)

Thus, by Barratt – Priddy – Quillen: $B\mathcal{C}ob_0 \simeq \Omega^{\infty-1} \mathbb{S}$

sphere spectrum

Idea: Find for each $d \geq 0$ the correct spectrum!

- Recall that in Michael's talk, we considered the Thom spectrum MO and obtained for each embedding $W^d \hookrightarrow \mathbb{R}^{d+n}$ an element in
$$\Omega^{d+n} MO_n$$
by the Pontrjagin – Thom construction, followed by the classifying map for the normal bundle ν.
- In order to get a spectrum for a fixed dimension d, the tautological bundle should "know" the dimension of its complement (which is $\dim TW = d$)

Constr: (Affine Thom spectrum)

- Consider on $\mathbb{R}^0 \subseteq \mathbb{R}^1 \subseteq \cdots \subseteq \mathbb{R}^\infty$ the standard inner product.
- Over the $G_{d,n} := Gr_d(\mathbb{R}^{d+n})$, we have the affine tautological bundle

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & \gamma^\perp_{d,n} = \{ (v, V) \in \mathbb{R}^{d+n} \times G_{d,n} ;\ v \in V \} \arrow[d] \\ & G_{d,n} \end{tikzcd}$$

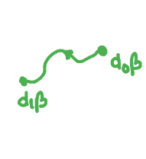
(this is merely convention; we could have taken γ over $Gr_n(\mathbb{R}^{d+n})$)

- $\mathbb{R}^{d+n} = \mathbb{R}^{d+n} \times \{0\} \hookrightarrow \mathbb{R}^{d+n+1}$ induces $\iota_n : G_{d,n} \longrightarrow G_{d,n+1}$ and
$$\iota_n^* \gamma^\perp_{d,n+1} = \gamma^\perp_{d,n} \oplus \underline{\mathbb{R}}$$
- We define the affine d-dim Thom spectrum $MTO(d)$ by
$$MTO(d)_{d+n} := Th(\gamma^\perp_{d,n})$$
and with structure maps
$$\Sigma\, MTO(d)_{d+n} = Th(\gamma^\perp_{d,n}) \wedge S^1 \cong Th(\gamma^\perp_{d,n} \oplus \underline{\mathbb{R}}) \longrightarrow Th(\gamma^\perp_{d,n+1}) = MTO(d)_{d+n+1}$$

Constr: We sketch a map $\alpha : B\mathcal{C}ob_d \longrightarrow \Omega^{\infty-1} MTO(d)$:

① For each space X, we consider the path category PX:

- $ob(PX) = \mathbb{R} \times X$
- $mor(PX) = \left\{ (a_0, a_1, \beta) ;\ \begin{smallmatrix} a_0 \leq a_1 \\ \beta: [a_0, a_1] \to X \end{smallmatrix} \right\} \subseteq \mathbb{R}^2 \times X^{\mathbb{R}}$



- $d_k(a_0, a_1, \beta) = (a_k, \beta(a_k))$
  $s_0(a, x) = (a, a, const_x)$
- composition = "concatenation" of paths

Then $X \simeq \mathbb{R} \times X \xrightarrow{\text{0-simplices}} BPX$ is a weak equivalence.

② Let $W \hookrightarrow [a_0, a_1] \times \mathbb{R}^{d+n-1}$ be an embedded cobordism.

- The normal bundle $\mathbb{R}^n \longrightarrow \nu \longrightarrow W$ can be written as a pullback

$$\begin{tikzcd} \nu \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma^\perp_{d,n} \arrow[d] \\ W \arrow[r] & G_{d,n} \end{tikzcd}$$

- The Pontrjagin – Thom construction gives rise to
$$[a_0, a_1]_+ \wedge S^{d+n-1} \longrightarrow Th(\nu)$$
- The adjoint of the composition is a path
$$[a_0, a_1] \longrightarrow \Omega^{d+n-1} MTO(d)_{d+n}$$

③ Letting $n \longrightarrow \infty$, we get maps
$$\mathrm{Emb}(W) \longrightarrow mor(P\Omega^{\infty-1} MTO(d))$$
which assemble into a functor $\mathcal{C}ob_d \longrightarrow P\Omega^{\infty-1} MTO(d)$

④ Applying B, we get
$$\alpha : B\mathcal{C}ob_d \longrightarrow BP\Omega^{\infty-1} MTO(d) \simeq \Omega^{\infty-1} MTO(d).$$

Thm: (GMTW, main thm) α is a weak equivalence:
$$B\mathcal{C}ob_d \simeq \Omega^{\infty-1} MTO(d)$$

Expl: $MTO(0)_n = Th(\mathbb{R}^n_*) = S^n$, i.e. $MTO(0) = \mathbb{S}$

$\rightarrow B\mathcal{C}ob_0 \simeq \Omega^{\infty-1} \mathbb{S}$ (recovers BPQ)

4 Application: the Madsen – Weiss thm and the Mumford conjecture

Upshot: GMTW for $d=2$ ⟹ Madsen – Weiss ⟹ Mumford conjecture

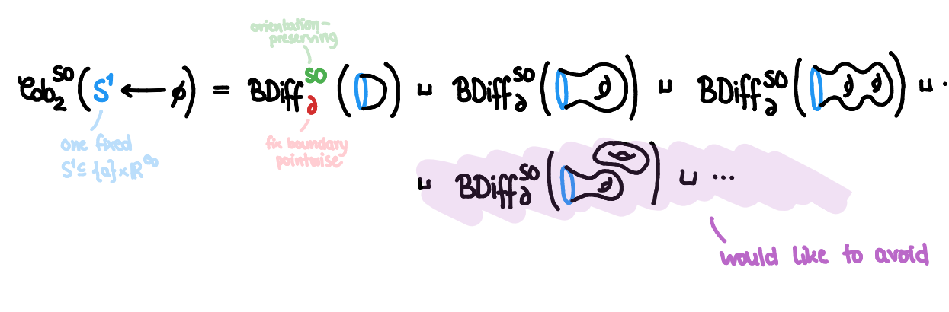
$\Omega B\left(\coprod_{g \geq 0} \mathcal{M}_{g,1}\right) \simeq \Omega^\infty MTSO(2)$

$H^*(\mathcal{M}_{\infty,1}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots], \ |\kappa_i| = 2i$

Step 1: (Tangential structures) As in Michael's talk:

- A tangential structure is a fibration $\vartheta : B \longrightarrow Gr_d(\mathbb{R}^\infty)$ (here we only need it for one fixed dimension)
  - get fibration $B_{d,n} \longrightarrow G_{d,n}$ by pullback
  - get bundle $\gamma^{\vartheta,\perp}_{d,n} \longrightarrow B_{d,n}$ again by pullback
- Given a tangential structure ϑ, we consider:
  - the category $\mathcal{C}ob^\vartheta_d$ of cobordisms with ϑ-structure,
  - the affine Thom spectrum with ϑ-structures $MT\vartheta(d)$.
- The main thm generalises to $B\mathcal{C}ob^\vartheta_d \simeq \Omega^{\infty-1} MT\vartheta(d)$.

Expl: Orientation is a ϑ-structure $BSO(d) \longrightarrow BO(d)$ for each $d \geq 0$. We get $\mathcal{C}ob^{SO}_d$ and $MTSO(d)$. Note



$\mathcal{C}ob^{SO}_2(S^1 \leftarrow \emptyset) = B\mathrm{Diff}^{SO}_\partial(\ ) \sqcup B\mathrm{Diff}^{SO}_\partial(\ ) \sqcup B\mathrm{Diff}^{SO}_\partial(\ ) \sqcup \cdots \sqcup B\mathrm{Diff}^{SO}_\partial(\ ) \sqcup \cdots$

one fixed $S^1 \subseteq \{a\} \times \mathbb{R}^\infty$

orientation-preserving

fix boundary pointwise

would like to avoid

Step 2: (Positive boundary subcategory)
Consider the subcategory $\mathcal{C}ob^\vartheta_{d,\partial} \hookrightarrow \mathcal{C}ob^\vartheta_d$ containing only cobordisms W with:

Each component of W touches $\partial_0 W$.

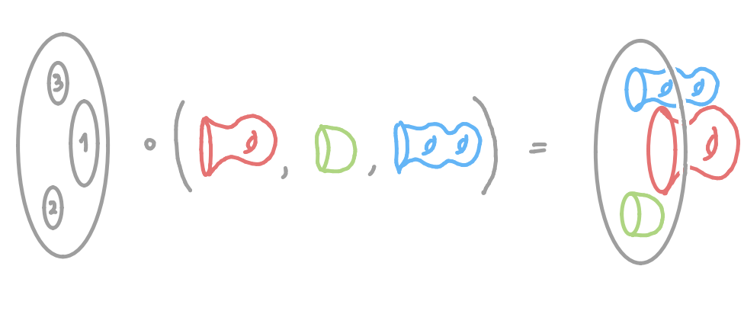
Expl: $\mathcal{C}ob^{SO}_{2,\partial}(S^1 \leftarrow \emptyset) = \coprod_{g \geq 0} \underbrace{B\mathrm{Diff}^{SO}_\partial(\Sigma_{g,1})}_{\mathcal{M}_{g,1}}$ (moduli space of surfaces of type $\Sigma_{g,1}$)

Thm: (GMTW 6.1, quite hard)
For $d \geq 2$ and any tangential structure,
$$B\mathcal{C}ob^\vartheta_{d,\partial} \longrightarrow B\mathcal{C}ob^\vartheta_d \text{ is a weak equivalence.}$$

Cor: $B\mathcal{C}ob^{SO}_{2,\partial} \simeq \Omega^{\infty-1} MTSO(2)$.

Rmd: (Moduli spaces of surfaces)

① The collection $\mathcal{M}_{\bullet,1} = \coprod_{g \geq 0} \mathcal{M}_{g,1}$ is an E_2-algebra by



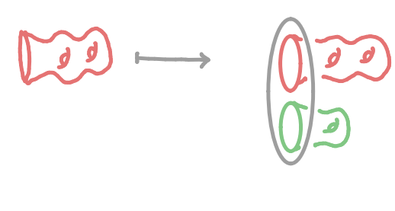
② The canonical map $\mathcal{M}_{\bullet,1} \longrightarrow \Omega B\mathcal{M}_{\bullet,1}$ is a group completion, i.e.

$$\begin{tikzcd} H_*(\mathcal{M}_{\bullet,1}) \arrow[r] \arrow[d] & H_*(\Omega B\mathcal{M}_{\bullet,1}) \\ H_*(\mathcal{M}_{\bullet,1})[\pi_0^{-1}] \arrow[ur, dashed, "\cong"'] & \end{tikzcd}$$

(and this determines $\Omega B\mathcal{M}_{\bullet,1}$ among simple spaces up to weak equivalence)

③ Fixing $s \in \mathcal{M}_{1,1}$, we get a stabilisation map

$$- \cdot s : \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g+1,1}$$



Define the stable moduli space

$$\mathcal{M}_{\infty,1} := \mathrm{hocolim}\left(\mathcal{M}_{0,1} \xrightarrow{-\cdot s} \mathcal{M}_{1,1} \xrightarrow{-\cdot s} \cdots \right)$$

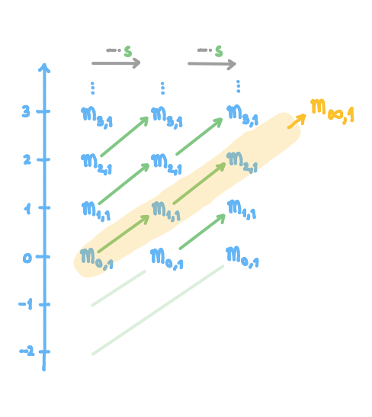
④ One way of expressing the group completion is by "inverting $s \in \mathcal{M}_{1,1}$":

$$\mathrm{hocolim}\left(\mathcal{M}_{\bullet,1} \xrightarrow{-\cdot s} \mathcal{M}_{\bullet,1} \xrightarrow{-\cdot s} \cdots \right) \simeq \mathcal{M}_{\infty,1} \times \mathbb{Z}$$

Indeed: $H_*(\mathcal{M}_{\infty,1} \times \mathbb{Z}) \cong H_*(\Omega B\mathcal{M}_{\bullet,1})$. Thus:

$$H_*(\mathcal{M}_{\infty,1}) \cong H_*(\Omega_0 B\mathcal{M}_{\bullet,1})$$

path component of the constant loop



Q: How to relate $\mathcal{M}_{\infty,1}$ with $B\mathcal{C}ob^{SO}_{2,\partial}$?

Rmk: Let $\mathcal{C}$ be a top. category and $x, y \in ob(\mathcal{C})$. Then there is a map

$$\mathcal{C}(y \leftarrow x) \longrightarrow \Omega_{y \leftarrow x} B\mathcal{C}$$

paths from the vertex x to the vertex y

which sends each f to the path along the 1-simplex of f.

Expl: Have map

$$\varphi : \mathcal{M}_{\bullet,1} = \mathcal{C}ob^{SO}_{2,\partial}(S^1 \leftarrow \emptyset) \longrightarrow \Omega_{S^1 \leftarrow \emptyset} B\mathcal{C}ob^{SO}_{2,\partial} \simeq \Omega B\mathcal{C}ob^{SO}_{2,\partial}$$

Prop: (GMTW 7.1, doable)
φ is a group completion!

Cor: (Madsen – Weiss 2007)

$$\Omega B\mathcal{M}_{\bullet,1} \simeq \Omega^\infty MTSO(2)$$

Cor: (Mumford conjecture)

$$H^*(\mathcal{M}_{\infty,1}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \qquad |\kappa_i| = 2i$$

Pf: We know $H^*(\mathcal{M}_{\infty,1}) \cong H^*(\Omega^\infty_0 MTSO(2))$
Now use standard methods to calculate the right side rationally.
(e.g. Hatcher: "A short exposition on the Madsen – Weiss thm", Apx. C1)